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A NOTE ON STRICTLY CYCLIC WEIGHTED SHIFTS

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ABSTRACT. We answer negatively Shields' Question 15 in [1].

The object of this note is to answer negatively one of the many questions raised by Allen L. Shields in his article [1]. Question 15 of [1] is: If T is a unilateral weighted shift with weights w_n decreasing to 1 and if $\sum_{n=0}^{\infty} \beta(n)^{-2} < \infty$ where $\beta(n) = w_0 w_1 \cdots w_{n-1}$, must T be strictly cyclic? If $\{w_n\}$ decreases to 0, must T be strictly cyclic? Strongly strictly cyclic?

By Proposition 32 of [1, p. 96], to answer the question negatively it will suffice to show that the supremum

$$\operatorname*{Sup}_n \sum_{k=0}^{n} \left(\frac{\beta(n)}{\beta(k)\beta(n-k)} \right)^2 \tag{1}$$

is infinite.

For the first part of the question, we let $\{a_j\}$ be any sequence decreasing to 1 and we choose an increasing sequence of positive integers so that

$$\left[a_1^{n_1} a_2^{n_2 - n_1} \cdots a_j^{n_j - n_{j-1}} \right]^{-2} \left(\frac{a_{j+1}^2}{a_{j+1}^2 - 1} \right) < 2^{-j} \tag{2}$$

and

$$(1 + n_j) \left(\frac{a_j}{a_1} \right)^{2n_{j-1}} > j. \tag{3}$$

Such a choice is clearly possible. We set $w_n = a_1$, if $0 \leqslant n < n_1$, and we set $w_n = a_j$, if $n_{j-1} \leqslant n < n_j$. In order to verify that our selection works we use (2) to show that $\sum_{n \geqslant 0} \beta(n)^{-2} < a_1^2/(a_1^2 - 1) + 1$ and (3) to show that the supremum (1) is infinite.

Consider (2) first. We have

$$\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} = \sum_{n=0}^{n_1} \frac{1}{\beta(n)^2} + \sum_{j=2}^{\infty} \left(\sum_{n_{j-1} < n \leqslant n_j} \frac{1}{\beta(n)^2} \right) \tag{4}$$

where $\beta(n)$ is given by the equations $\beta(n) = a_1^n$ if $n \leqslant n_1$, and $\beta(n) = a_1^{n_1} a_2^{n_2 - n_1} \cdots a_j^{n - n_{j-1}}$ if $n_{j-1} < n \leqslant n_j$.

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Therefore, $\sum_{n=0}^{n_1} \beta(n)^{-2} < \sum_{n=0}^{\infty} a_1^{-2n} = a_1^2/(a_1^2 - 1)$, while

$$\sum_{n_{j-1}<n<n_j} \frac{1}{\beta(n)^2} = \left[a_1^{n_1} a^{n_2-n_1} \cdots a_{j-1}^{n_{j-1}-n_{j-2}}\right]^{-2} \left(\sum_{n_{j-1}<n<n_j} \frac{1}{a_j^{2(n-n_{j-1})}} \right)$$

$$< \left[a_1^{n_1} \cdots a_{j-1}^{n_{j-1}-n_{j-2}}\right]^{-2} \left[\frac{a_j^2}{a_j^2 - 1} \right] < 2^{-j+1}.$$

This with (4) shows that $\sum_{n=0}^{\infty} \beta(n)^{-2} < a_1^2/(a_1^2 - 1) + 1$.

To show that (1) is violated we show that a lower bound for

$$\sum_{k=0}^{n_j} \left(\frac{\beta(n_j)}{\beta(k)\beta(n_j - k)} \right)^2 \tag{5}$$

is j. Consider

$$\frac{\beta(n_j)}{\beta(k)\beta(n_j - k)} = \frac{w_k \cdots w_{n_j - 1}}{w_0 \cdots w_{n_j - (k+1)}};$$

in both top and bottom there are no more than n_{j-1} weights different from a_j. Thus, we can say that $\beta(n_j)/\beta(k)\beta(n_j - k) > (a_j/a_1)^{n_{j-1}}$, which together with (3) gives (5).

Note that it is actually possible to choose the sequence $\{w_n\}$ strictly decreasing if desired. One might ask if it is possible to choose the weight sequence to be convex.

The second part of the question is answered with a similar construction. Simply choose $\{a_j\}$ decreasing to zero and choose $\{n_j\}$ so that (3) is satisfied. If $\{w_n\}$ is defined as before, then the supremum (1) is infinite, and T is not strictly cyclic.

References

1. A. L. Shields, *Weighted shift operators and analytic function theory*, Math. Surveys No. 13, Amer. Math. Soc., Providence, R.I., 1974.

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