A SUFFICIENT CONDITION FOR LINEAR GROWTH OF VARIANCES IN A STATIONARY RANDOM SEQUENCE

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Abstract. Suppose \((X_k, k = \ldots, -1, 0, 1, \ldots)\) is a weakly stationary random sequence. For each positive integer \(n\) let \(S_n = X_1 + \cdots + X_n\) and let \(\tau(n) = \sup \{|\text{Corr}(\sum_{k=-l}^{m} \sum_{k=m}^{n} X_k)|: m > n, l > 0\}\). If \(\text{Var} S_n \to \infty\) as \(n \to \infty\) and \(\sum_{n=0}^{\infty} \tau(2^n) < \infty\), then \(n^{-1} \text{Var} S_n\) converges to a finite positive limit as \(n \to \infty\). A bound on the rate of convergence is obtained.

Let \((X_k, k = \ldots, -1, 0, 1, \ldots)\) be a weakly stationary sequence of random variables on a probability space \((\Omega, \mathcal{F}, P)\). That is, \(EX_k^2 < \infty\) \(\forall k\), there are constants \(\mu\) and \(\gamma(0)\) such that \(EX_k = \mu\) and \(\text{Var} X_k = \gamma(0)\) \(\forall k\), and there is a sequence of numbers \(\gamma(1), \gamma(2), \gamma(3), \ldots\) such that \(\text{Cov}(X_k, X_l) = \gamma(|k - l|)\) \(\forall k \neq l\). For \(-\infty < J < L < \infty\) let \(H^2_{J,L}\) denote the \(L^2\)-closure of the linear space spanned by the r.v.'s \(X_k - \mu, J < k < L\). For \(n = 1, 2, 3, \ldots\) let \(S_n = X_1 + X_2 + \cdots + X_n\) and define the quantities

\[
r(n) = \sup \{|\text{Corr}(f, g)|: f \in H^0_{-\infty, \infty}, g \in H^\infty_n\},
\]

\[
\tau(n) = \sup_{m \geq n} \left| \text{Corr} \left( \sum_{k=-l}^{m} X_k, \sum_{k=m}^{n} X_k \right) \right|.
\]

Obviously the sequences \(\{r(n)\}\) and \(\{\tau(n)\}\) are each nonincreasing, and \(0 < \tau(n) < r(n)\).

Ibragimov and Rozanov [3, Note 2, p. 190] proved the following theorem (it was originally proved in [4]).

**Theorem 0 (IBRAGIMOV AND ROZANOV).** If \((X_k)\) is weakly stationary and \(\sum_{n=0}^{\infty} \tau(2^n) < \infty\), then \((X_k)\) has an absolutely continuous spectral distribution function, with a continuous spectral density \(f(\lambda)\).

In [3, Note 2, p. 190] this is stated for stationary Gaussian sequences, but the extension to general weakly stationary sequences is easy. Ibragimov [2, Theorem 2.2] proved a central limit theorem for strictly stationary random sequences satisfying a condition similar to but stronger than \(\sum \tau(2^n) < \infty\); in that theorem it was assumed that \(f(0) > 0\). In central limit theorems for weakly dependent random variables it is common practice to assume \(\text{Var} S_n \to \infty\) as \(n \to \infty\). Consider the question whether \(f(0) > 0\) follows if one assumes \(\sum \tau(2^n) < \infty\) and \(\text{Var} S_n \to \infty\).
The answer is affirmative, and one can prove this as a corollary to Theorem 0 by using the results in Chapters 4 and 5 of [3]. It also follows easily from the following theorem.

**THEOREM 1.** If \((X_k)\) is weakly stationary, \(\text{Var} \ S_n \to \infty\) as \(n \to \infty\), and \(\sum_{n=0}^{\infty} \tau(2^n) < \infty\), then the following statements hold:

(i) There exists a finite positive constant \(\sigma^2\) such that \(\lim_{n \to \infty} n^{-1} \text{Var} \ S_n = \sigma^2\).

(ii) For each \(\gamma, 0 < \gamma < 1\), there exist positive constants \(C\) and \(D\) such that for all \(m > 1\),

\[
|\sigma^2 - m^{-1} \text{Var} \ S_m| < C \left( m^{(\gamma - 1)/2} + \sum_{n=0}^{\infty} \tau\left(\frac{\left(2^m\right)^\gamma}{D}\right) + 1 \right)
\]

where \([t]\) denotes the greatest integer \(< t\).

Theorem 1 covers the case where \(\tau(n) = o((\log n)^{-1+\varepsilon})\) for some \(\varepsilon > 0\). For the stationary Gaussian sequence given in Example 1 on pp. 179–180 of [3], \(\tau(n) = O((\log n)^{-1})\) and \(n^{-1} \text{Var} \ S_n \to \infty\) as \(n \to \infty\). Berkes and Philipp [1, Theorem 4] prove an “almost sure invariance principle” for strictly stationary random sequences satisfying the \(\phi\)-mixing condition with a rather slow (logarithmic) mixing rate. Their conditions satisfy the hypothesis of Theorem 1 (see [1, Lemma 4.1.1, p. 44]), and it would be interesting to see if one can choose \(a_N = \sigma^2 N\) in the conclusion of their theorem. In the cases where \(\tau(n) \to 0\) rapidly, other methods generally give much better bounds on \(|\sigma^2 - m^{-1} \text{Var} \ S_m|\) than Theorem 1(ii) does.

**Proof of Theorem 1.** We will assume \(E X_k = 0\) and \(\text{Var} \ X_k = 1\). Let \(S_0 = 0\). For any r.v. \(X\) let \(||x|| = (EX^2)^{1/2}\). For each \(n = 1, 2, 3, \ldots\) let \(g(n) = n^{-1/2} ||S_n||\).

**Lemma 1.** If \(N > 1, K > 0,\) and \(\tau(K + 1) < \frac{1}{2}\), then for \(L = 2N\) and for \(L = 2N + 1\) one has

\[
||S_N|| \cdot \left[2\left(1 - \tau(K + 1)\right)\right]^{1/2}(1 - 2b/||S_N||) < ||S_L||
\]

\[
< ||S_N|| \cdot \left[2\left(1 + \tau(K + 1)\right)\right]^{1/2}(1 + 2b/||S_N||)
\]

where \(b \equiv \max\{||S_{K-1}||, ||S_K||\}\).

**Proof.**

\[
S_L = S_N + (S_{N+K} - S_N) + (S_{2N+K} - S_{2N}) - (S_{2N+K} - S_L),
\]

\(2(1 - \tau(K + 1))||S_N||^2 < ||S_N + S_{2N+K} - S_{N+K}||^2 < 2(1 + \tau(K + 1))||S_N||^2\)

and Lemma 1 follows from Minkowski's inequality.

**Lemma 2.** Given any \(\varepsilon > 0\) and any positive integer \(L_0\), there are positive integers \(N\) and \(L\) with \(L > L_0\) such that \(\forall \, l, L < l < 2L\), one has

\[
(1 - \varepsilon)g(N) \leq g(l) \leq (1 + \varepsilon)g(N).
\]

**Proof.** For each integer \(J \geq 2\), \(\text{Corr}(S_N, S_{J(N+1)}) \to 0\) as \(N \to \infty\). (For \(J = 2\) this just follows from (1).) Hence \(\forall \, J \geq 2, g(NJ)/g(N) \to 1\) as \(N \to \infty\).
Let $M$ be a positive integer such that $(1 - \varepsilon) < (1 - \varepsilon)^{1/2} - M^{-1/2} < (1 + \varepsilon)^{1/2} + M^{-1/2} < (1 + \varepsilon)$. Let $N$ be such that $MN > L_0$, 

$$\|S_N\| = \max\{\|S_k\|, 1 \leq k < N\},$$

and for each $m, M < m < 2M$, one has

$$(1 - \varepsilon)^{1/2} < \frac{g(mN)}{g(N)} < (1 + \varepsilon)^{1/2}.$$ 

Let $L = MN$.

If $L < l < 2L$, then for some $m$ one has $M < m < 2M - 1$ and $mN < l < (m + 1)N$, and hence

$$\begin{align*}
(1 - \varepsilon)g(N) &< \left[ (1 - \varepsilon)^{1/2} - (m + 1)^{-1/2} \right] g(N) \\
&< \left[ (m + 1)N \right]^{-1/2} \left( \|S_{(m+1)N}\| - \|S_N\| \right) \\
&< g(l) < \left[ (mN)^{-1/2} \|S_{mN}\| + \|S_N\| \right] \\
&< \left[ (1 + \varepsilon)^{1/2} + m^{-1/2} \right] g(N) < (1 + \varepsilon)g(N)
\end{align*}$$

and Lemma 2 is proved.

Let $0 < A < 1$ be sufficiently small that if $(a_n, n = 1, 2, \ldots)$ is any sequence of real numbers such that $\Sigma_n |a_n| < A$ then $|1 - \Pi_n (1 + a_n)| < 2\Sigma_n |a_n|$ and $|1 - \Pi_n (1 + a_n)^{-1}| < 2\Sigma_n |a_n|$. Let $[\ ]$ denote the greatest-integer function.

Proof of Theorem 1(i). Assume $0 < \varepsilon < A$. Let $N$ be a positive integer such that $\Sigma_{n=0}^{\infty} \tau(2^{n+n/6}) < \varepsilon/6$. Let $C > 0$ be such that $\Sigma_{n=0}^{\infty} 2^{n+1-n/6}C^{-1} < \varepsilon/6$. Let $L_0$ be a positive integer such that (i) $\forall J > L_0, \|S_J\| > C$, (ii) $\Sigma_{n=0}^{\infty} 2^{-n}L_0^{-1} < \varepsilon/6$, and (iii) $\forall J > L_0, g(2J) > 2^{-1/6}g(J)$ and $g(2J + 1) > 2^{-1/6}g(J)$ (see Lemma 1). For each $n = 0, 1, 2, \ldots$ let $K_n = [2^{n+n/6}]$. Using Lemma 2, let the positive integers $H$ and $L$ be such that $L > L_0$ and $\forall l, L < l < 2L$, one has $(1 - \varepsilon)g(H) < g(l) < (1 + \varepsilon)g(H)$.

Let $m$ be an arbitrary positive integer satisfying $m > 2L$. We wish to prove

$$(1 - \varepsilon)^{1/2}g(H) < g(m) < (1 + \varepsilon)^{1/2}g(H).$$

For some positive integer $M, 2^ML < m < 2^{M+1}L$. There is a sequence of positive integers $J_0, J_1, \ldots, J_M$ such that $m = J_M, L < J_0 < 2L$, and for each $n = 0, 1, \ldots, M - 1, J_{n+1} = 2J_n$ or $J_{n+1} = 2J_n + 1$.

For each $n = 0, 1, \ldots, M - 1$, one has $\|S_{J(n)}\| > 2^{n/3}\|S_{J(0)}\| > 2^{n/3}C$, and using Lemma 1 and the inequality $\|S_K\| < K$ one has

$$\begin{align*}
g(J_{n+1}) &> g(J_n)(2J_n/J_{n+1})^{1/2}(1 - \tau(K_n + 1))^{1/2}(1 - 2K_n/ (2^{n/3}C)) \\
&> g(J_n)(1 - 2^{-n}L_0^{-1})^{1/2}(1 - \tau(K_n + 1))^{1/2}(1 - 2^{n+1-n/6}C^{-1}),
\end{align*}$$

$$\begin{align*}
g(J_{n+1}) &< g(J_n)(1 + \tau(K_n + 1))^{1/2}(1 + 2K_n/ (2^{n/3}C)) \\
&< g(J_n)(1 + \tau(K_n + 1))^{1/2}(1 + 2^{n+1-n/6}C^{-1}).
\end{align*}$$

Hence

$$\begin{align*}
g(m) > g(J_0) \prod_{n=0}^{M-1} \left[ (1 - 2^{-n}L_0^{-1})^{1/2}(1 - \tau(K_n + 1))^{1/2}(1 - 2^{n+1-n/6}C^{-1}) \right],
\end{align*}$$

$$\begin{align*}
g(m) < g(J_0) \prod_{n=0}^{M-1} \left[ (1 + \tau(K_n + 1))^{1/2}(1 + 2^{n+1-n/6}C^{-1}) \right].
\end{align*}$$
Since $\epsilon < A$ we get $(1 - \epsilon)g(J_0) < g(m) < (1 + \epsilon)g(J_0)$ and hence $(1 - \epsilon)^2g(H) < g(m) < (1 + \epsilon)^2g(H)$, which is what we wanted to prove.

Hence $\lim \inf g(n)$ and $\lim \sup g(n)$ are finite positive numbers, and their ratio can be forced arbitrarily close to 1 if $\epsilon$ is chosen sufficiently small. Theorem 1(i) follows.

**Proof of Theorem 1(ii).** Let $\sigma^2$ be as in Theorem 1(i). Assume $0 < \gamma < 1$. There are constants $0 < C_1 < C_2$ such that, for all $n = 1, 2, 3, \ldots$, $C_1 < g(n) < C_2$.

Let $N$ be a positive integer such that $\sum_{n=0}^{\infty} \tau([2^{(N+n)\gamma}]) < A/3$. Let $L$ be a positive integer such that $\sum_{n=0}^{\infty} 2^{1+N+(n-1)/2}C_2/(C_1L) < A/3$.

Suppose $M$ is a nonnegative integer and $2^ML^2 < m < 2^{M+1}L^2$. For each $n = 0, 1, 2, \ldots$ let $J_n = 2^nm$ and let $K_n = [2^{(N+n)\gamma}]$. Then by Lemma 1,

$$g(J_{n+1}) > g(J_n)(1 - \tau(K_{n+M} + 1))^{1/2}(1 - 2C_2K_n^{1/2}/(C_1J_n^{1/2})),$$

$$> g(J_n)(1 - \tau(K_{n+M} + 1))^{1/2}(1 - 2C_1(1 - L/2)(1 + \tau(K_{n+M} + 1))^{1/2}/(C_1J_n^{1/2})),$$

and hence by the definition of $A$,

$$|\sigma^2 - g^2(m)| \leq 2\sigma^2 \sum_{n=0}^{\infty} 2^{1+N}C_2/(C_1L) .$$

If we let $M \to \infty$ and $m \to \infty$ subject to the restriction $2^ML^2 < m < 2^{M+1}L^2$, then we get the following:

$$\sum_{n=0}^{\infty} 2^{(M+n)(\gamma-1)/2} = O(2^{M(\gamma-1)/2}) = O(m^{(\gamma-1)/2}),$$

$$\sum_{n=0}^{\infty} \tau(K_{n+M} + 1) \leq \sum_{n=0}^{\infty} \tau([2^nm^{1/2}]/L^2) + 1).$$

Theorem 1(ii) follows.

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**References**


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