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COMPACT COVARIANCE OPERATORS[1]

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ABSTRACT. Let B be a real separable Banach space and R: $B^* \to B$ a covariance operator. All representations of R in the form $\Sigma e_n \otimes e_n$, $\{e_n, n \geq 1\} \subset B$, are characterized. Necessary and sufficient conditions for R to be compact are obtained, including a generalization of Mercer's theorem. An application to characteristic functions is given.

1. Introduction. The study of covariance operators is a major component in the theory of probability measures on Banach spaces [10], [9], [1]. The covariance operator of a strong second-order measure is always compact [2]; however, the covariance operator of a weak second-order measure need not be compact. In this paper we first characterize series representations of covariance operators, and then give a set of necessary and sufficient conditions for a covariance operator to be compact. The classical Mercer's theorem [7] can be obtained as an immediate corollary. These results are then applied to extend a result of Prohorov and Sazanov [6] on relative compactness of probability measures from Hilbert space to Banach space.

2. Definitions and notation. B is a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*. A linear operator R: $B^* \to B$ is a covariance operator if R is symmetric and nonnegative: $\langle Ru, v\rangle = \langle u, Rv\rangle$ and $\langle Ru, u\rangle \geq 0$, for all u, v in B^*. A probability measure μ on the Borel σ-field of B is said to be weak second-order if $\int_B \langle x, u\rangle^2 \, d\mu(x) < \infty$, for all u in B^*; μ is strong second-order if $\int_B \|x\|^2 \, d\mu(x) < \infty$. Every weak second-order measure μ has a mean element m in B and a covariance operator R: $B^* \to B$ [9], [10], defined by

$$\langle m, v\rangle = \int_B \langle x, v\rangle \, d\mu(x), \qquad \langle Ru, v\rangle = \int_B \langle x - m, u\rangle\langle x - m, v\rangle \, d\mu(x),$$

for all u, v in B^*. Strong second-order measures have compact covariances; the strong second-order property is not necessary in order that μ have compact covariance.

For a covariance operator R: $B^* \to B$ it is well known [8], [1], that there exists a separable Hilbert space $H \subset B$ such that the natural injection j: $H \to B$ is continuous and $R = jj^*$. H is the RKHS of R and is the completion of range(R) with respect to the inner product $\langle \cdot, \cdot \rangle_H$ defined by $\langle Ru, Rv\rangle_H = \langle Ru, v\rangle$.

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I_H will denote the identity on H. For u, v in B, z in B^* (resp. in H), $(u \otimes v)(z) = \langle v, z\rangle u$ (resp., $\langle v, z\rangle_H u$). If T is any map $r(T) \equiv \text{range}(T)$. τ_c is the linear topology on B^* determined by a neighborhood base at zero of the form $V_{C,\varepsilon}(0) = \{f \in B^*: \sup_{x\in C}\langle f, x\rangle^2 < \varepsilon\}$ for all $\varepsilon > 0$ and all compact sets $C \subset B$ (τ_c is the topology of uniform convergence on compact sets). For a given covariance operator R: $B^* \to B$, q_R is the real-valued quadratic functional on B^* defined by $q_R f = \langle Rf, f\rangle$. The notation $R = \Sigma_n e_n \otimes e_n$ for $\{e_n, n \geqslant 1\} \subset B$ means that the sequence $(\Sigma_1^N e_n \otimes e_n)$ converges to R in the strong operator topology: $\Sigma_1^N\langle e_n, f\rangle e_n \to Rf$ in the norm topology of B, for all f in B^*. $I_H = \Sigma_n e_n \otimes e_n$ has a similar interpretation. If $\{u_n, n \geqslant 1\}$ is any orthonormal basis for H, then $R = \Sigma\, ju_n \otimes ju_n$ [9]. K_R will denote the unit ball in H.

If μ is a probability measure on the Borel σ-field of B, its characteristic functional $\hat{\mu}$ is defined as $\hat{\mu}(x) = \int_B e^{i\langle x,y\rangle}\, d\mu(y)$, for x in B^*.

3. Representation of covariance operators. In this section, R is an arbitrary covariance operator.

THEOREM 1. *$R = \Sigma_n e_n \otimes e_n$ if and only if $e_n = jv_n$, $v_n \in H$ for $n \geqslant 1$, and $I_H = \Sigma_n v_n \otimes v_n$.*

PROOF. It suffices to show that the stated conditions are necessary for $R = \Sigma_n e_n \otimes e_n$. Suppose $R = \Sigma_n e_n \otimes e_n$, and fix e_k. Let $P_k = e_k \otimes e_k$. To show $e_k \in \text{range}(j)$, let (as in [3]) D: $r(j^*) \to B$ be defined by $Dj^*f = P_k f$. Then $\|Dj^*f\|^2 = \|P_k f\|^2 = \|e_k\|^2\langle e_k, f\rangle^2 \leqslant \|e_k\|^2\Sigma_n\langle f, e_n\rangle^2 = \|e_k\|^2\langle Rf, f\rangle = \|e_k\|^2\|j^*f\|_H^2$. Thus D can be extended to a continuous linear map from $\overline{r(j^*)} = H$ into B. From its definition, $Dj^* = P_k$, so $P_k = jD^*$ and thus $e_k \in \text{range}(j)$.

To see that $I_H = \Sigma v_n \otimes v_n$, where $jv_n = e_n$, $n \geqslant 1$, define $Q_N = \Sigma_1^N v_n \otimes v_n$. $Q_N = Q_N^*$ and $Q_N \geqslant 0$, so $Q_N^{1/2}$ exists. $\|Q_N^{1/2}j^*f\|_H^2 = \Sigma_1^N\langle f, e_n\rangle^2 \uparrow \|j^*f\|_H^2$, so that $\|Q_N^{1/2}\| \leqslant 1$ and $\|Q_N^{1/2}x\|_H \to \|x\|_H$ for all x in $r(j^*)$. Thus,

$$\left\|\sum_1^N (v_n \otimes v_n)j^*f - j^*f\right\|_H^2 = \|Q_N j^*f - j^*f\|_H^2$$
$$\leqslant -\|Q_N^{1/2}j^*f\|_H^2 + \|j^*f\|_H^2,$$

which converges to zero as $N \to \infty$ for any fixed f in B^*. Thus, $\Sigma v_n \otimes v_n = I_H$ on $r(j^*)$, and the result follows by $\overline{r(j^*)} = H$. □

REMARK. Suppose E is a locally convex topological vector space, R: $E' \to E$ is a covariance operator, and $R = jj^*$, where j: $H \to E$ is the injection and H is the RKHS of R. R will have such a representation, for example, if E is quasi-complete [8]. In this case, it is easily shown that Theorem 1 holds without modification.

The representation $I_H = \Sigma v_n \otimes v_n$ does not require that $\{v_n, n \geqslant 1\}$ be a CONS in H; however, sufficient conditions for $\{v_n, n \geqslant 1\}$ to be a CONS in H can be given.

PROPOSITION 1. *Suppose $I_H = \Sigma v_n \otimes v_n$; the following are equivalent:*

(1) $\|v_k\|_H = 1$,

(2) $v_k \notin \overline{\text{sp}\{v_n, n \neq k\}}$,

(3) $v_k \perp \overline{\text{sp}\{v_n, n \neq k\}}$.

If any of the above conditions holds for all $k \geqslant 1$, then $\{v_n, n \geqslant 1\}$ is a CONS for H.

4. Compact covariance operators.

THEOREM 2. *Suppose* $R = \Sigma e_n \otimes e_n$, $\{e_n, n \geqslant 1\} \subset B$. *Let* $\{v_n, n \geqslant 1\} \subset H$ *be such that* $e_n = jv_n$, $n \geqslant 1$. *The following are equivalent*:

(1) *R is compact*;

(2) *j is compact*;

(3) *$j[K_R]$ is compact in B*;

(4) *the series $\Sigma v_n \otimes jv_n$ converges uniformly in H on bounded subsets of B^**;

(5) *$(\Sigma_1^N e_n \otimes e_n)$ converges to R uniformly in B on bounded subsets of B^**;

(6) *q_R is w^*-continuous on bounded subsets of B^**;

(7) *q_R is τ_c-continuous.*

PROOF. (1) $\Rightarrow$ (2). Suppose $f_\alpha \to f$ in the w^* topology of B^*, where $\|f_\alpha\| \leqslant k$ for all α. Then $\|j^*f_\alpha - j^*f\|_H^2 = \langle R(f_\alpha - f), (f_\alpha - f)\rangle \leqslant 2k\|R(f_\alpha - f)\|_B$; since R is compact, $j^*f_\alpha \to j^*f$ in H [**4**, p. 486] and thus j is compact.

(2) $\Rightarrow$ (3). j compact implies $j[K_R]$ is relatively compact in B. Since K_R is weakly compact in H and j is weakly continuous, $j[K_R]$ is weakly compact in B, and thus closed.

(3) $\Rightarrow$ (2) by definition.

(2) $\Rightarrow$ (4). By Theorem 1, $\Sigma_n v_n \otimes v_n = I_H$. Set $Q_N = \Sigma_1^N v_n \otimes v_n$. If $A \subset B^*$ is bounded, then $\overline{j^*[A]}$ is compact; by Dini's theorem $\|Q_N^{1/2}x\|_H \uparrow \|x\|_H$ uniformly on $j^*[A]$. Hence $\|(Q_N - I)j^*x\|_H^2 \leqslant \|j^*x\|_H^2 - \|Q_N^{1/2}j^*x\|_H^2 \to 0$ uniformly on A.

(4) $\Rightarrow$ (5), since j is continuous.

(5) $\Rightarrow$ (1), since R is the uniform limit of compact operators.

(2) $\Leftarrow$ (6) follows from the fact that j is compact if and only if $j^*f_\alpha \to 0$ in the norm topology of H for all bounded generalized sequences (f_α) in B^* which are w^* convergent to zero [**4**, p. 486], and $q_R(f_\alpha) = \|j^*f_\alpha\|_H^2$.

(1) $\Rightarrow$ (7). Suppose R is compact. Writing $C = j[K_R]$, C is compact in B. $q_R(f) = \langle Rf, f\rangle = \|j^*f\|_H^2 = \sup_{x\in K}\langle j^*f, x\rangle_H^2 = \sup_{x\in C}\langle f, x\rangle^2$. Thus q_R is τ_c-continuous at zero. τ_c-continuity of q_R follows from $q_R(f_\alpha) = q_R(f_\alpha - f) - q_R(f) + 2\langle Rf, f_\alpha\rangle$.

(7) $\Rightarrow$ (1). Suppose q_R is τ_c-continuous. Using (6), R is compact if q_R is w^* continuous at 0 on bounded subsets of B^*. But B is separable so that the w^* topology on bounded subsets of B^* is metrizable and it suffices to consider sequences. Suppose $f_n \to^{w^*} 0$ and $\|f_n\| \leqslant k$. Let L be an arbitrary compact subset of B. Since (f_n) is bounded in B^* the f_n are equicontinuous and uniformly bounded as continuous functions on L. Thus, by the Arzela-Ascoli theorem [**4**, p. 266], (f_n) is relatively compact as a subset of $C^{\mathbf{R}}(L)$. Thus since $f_n \to^{w^*} 0$, f_n converges to 0 uniformly on L. Therefore $f_n \to^{\tau_c} 0$ and $q_R(f_n) \to 0$. This completes the proof of Theorem 2. $\square$

REMARKS. (1) Suppose r: $[0, 1] \times [0, 1] \to \mathbf{R}$ is continuous, symmetric and positive definite. For fixed $t \in [0, 1]$, let $\pi_t(x) = x_t$ for x in $C[0, 1]$; $\|\pi_t\| = 1$. A compact covariance operator R: $C^*[0, 1] \to C[0, 1]$ is defined by $[R\mu](t) = \int_0^1 r(t, s)\, d\mu(s)$ for any μ in $C^*[0, 1]$ (by Arzela-Ascoli theorem). Thus for $s, t \in [0, 1]$, $\langle R\pi_t, \pi_s\rangle = r(t, s)$. The integral operator in $L_2[0, 1]$ corresponding to the kernel r has continuous orthonormal eigenvectors $\{y_n, n \geqslant 1\}$ and associated

nonzero eigenvalues $\{\lambda_n, n \geqslant 1\}$; it is well known that $\{\lambda_n^{1/2} y_n, n \geqslant 1\}$ is a CONS in the RKHS H of R. Thus, from Theorem 2, $\sum_{n=1}^{N} \lambda_n y_n(t) y_n(s)$ converges uniformly to $r(t, s)$ for all t, s in [0, 1]. This is the classical Mercer's theorem [7, pp. 245–246].

(2) The fact that the unit ball of H is compact in B when R is compact was proved by Kuelbs [5] under the assumption that R is the covariance of a strong second-order measure.

5. Characteristic functionals. Let Λ denote a family of probability measures on B (separable Banach) and $\hat{\Lambda}$ the corresponding family of characteristic functionals.

THEOREM 3. *Let B be a separable Banach space. Then the following are equivalent*:

(a) *There exists a topology τ on B^* such that for each family Λ of probability measures on B, $\hat{\Lambda}$ is equicontinuous in this topology if and only if Λ is relatively compact in the topology of weak convergence.*

(b) *B is finite dimensional.*

PROOF. As in the Hilbert space case (see [6, Lemma 2]), τ_c is the weakest topology on B^* such that relative compactness of $\Lambda \Rightarrow$ equicontinuity of $\hat{\Lambda}$. Suppose that (a) holds. Then $\tau_c \subset \tau$ and τ_c-equicontinuity of $\hat{\Lambda}$ implies relative compactness of Λ. Now let R: $B^* \to B$ be any compact covariance operator. Let $\{e_n\}$ be a CONS in the RKHS of R. Define μ_k to be the zero mean Gaussian measure on B with covariance operator $\sum_1^k e_n \otimes e_n$. Then $\{\hat{\mu}_n\}$ is τ_c-equicontinuous by Theorem 2 and $\{\mu_k\}$ is relatively compact. Therefore R is the covariance of a Gaussian probability measure on B and, by [9, Theorem 11], B is finite dimensional. $\square$

Theorem 3 extends a result of Prohorov and Sazonov [6] who proved it for Hilbert spaces.

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