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AN EXAMPLE IN SHAPE THEORY

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ABSTRACT. We give an example of a compactum which cannot be "improved" within its shape class so that its shape theory agrees with its homotopy theory.

1. Introduction. S. Ferry [2] has proved the following theorem: For every $UV^{(1)}$ compactum X there is an "improved" compactum X' shape equivalent to X such that for every finite-dimensional compactum Z there is a one-to-one correspondence between the set of homotopy classes and the set of strong shape morphisms from Z to X'. Two questions arise concerning the hypotheses in that theorem:

(i) Is $UV^{(1)}$ a necessary requirement; i.e. is there a compactum that cannot be improved?

(ii) Is the finite dimensionality of Z essential?

In this paper we give an example to answer in the affirmative the first question.

I have to thank Ross Geoghegan for introducing me to Ferry's work and for his interest and help. In particular, he communicated a version of the folk lemma given in the Appendix. The referee suggested the version he wanted to see published and expressed regret Chapman had not done so in 1970. I am grateful to the referee for changing the original example (so that one would not be misled by the size of π_1), the Hawaiian earring.

2. The example. Let X_n denote the wedge of a circle with n copies of the k-sphere S^k, $n \geqslant 0$, $k \geqslant 2$. Let X be the limit of the sequence $X_0 \leftarrow X_1 \leftarrow \cdots$ with bond $p_m^n: X_n \to X_m$ equal to the identity on S^1 and the first m copies of S^k, and sending the remaining $n - m$ copies of S^k to the basepoint.

Note that X_n inherits a metric as a subset of $S^1 \vee B^{k+1} \subset S^1 \times B^{k+1}$, where B^{k+1} is the $(k+1)$-dimensional ball.

THEOREM. *For any compactum Y shape equivalent to X, the natural map*

$$[S^k, Y]_* \to \{\text{strong shape morphisms from } S^k \text{ to } Y\}_*$$

is not surjective.

To get an idea of the proof, consider first the case $Y = X$. Let $a_1, \ldots, a_n$ be the obvious generators of $\pi_k(X_n)$ as a module over $\pi_1(X_n) = \mathbf{Z}$ (i.e. a_i is represented by

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a map that sends S^k by the identity to the ith copy of S^k in X_n). We choose a generator t of $\pi_1(X_n)$ and denote the action of t on $\pi_k(X_n)$ by $g \mapsto tg$. Any infinite word of the form

$$a = \sum_{i=1}^{\infty} (t^{k_i} + t^{-k_i})a_i,$$

with $k_i \to \infty$ as $i \to \infty$, determines a shape morphism $S^k \to X$ which cannot be represented by a map. Since there is a forgetful map from strong shape morphisms to shape morphisms, this suffices to prove the theorem.

Let A denote the set of all such a's. It is the size of A that makes the theorem true. For, given any finite subset $A' \subset A$, there is a compactum X' and a shape equivalence h: $X \to X'$ such that for every $a \in A'$, ha is representable by a map: just take X' to be the shape mapping cylinder of $\coprod_{a \in A'} a$: $\coprod_{a \in A'} S^k \to X$.

PROOF OF THE THEOREM. Let Y be a compactum shape equivalent to X. It follows from the Appendix that there is a sequence of embeddings j_n^{n+1}: $X_{n+1} \times Q \to X_n \times Q$ (Q = Hilbert cube) such that

$$Y = \varprojlim\left(X_0 \times Q \xleftarrow{j_0^1} X_1 \times Q \xleftarrow{j_1^2} \cdots\right) = \bigcap_{n=0}^{\infty} j_0^n(X_n \times Q)$$

and such that j_n^{n+1} is homotopic to $p_n^{n+1} \times \mathrm{id}_Q$. The natural map from Y to $Y_n = X_n \times Q$ will be denoted by j_n.

If α: $S^k \to Y$ is a map, then $j_0\alpha = j_0^n j_n \alpha$: $S^k \to Y_0 = S^1 \times Q$ is null-homotopic and lifts to the universal cover $\mathbf{R}^1 \times Q$ of $S^1 \times Q$. We will construct a shape morphism ω: $S^k \to Y$ such that if ω_n: $S^k \to Y_n$ is any map representing the shape morphism $j_n\omega$, then the sequence $\operatorname{diam} \widetilde{j_0^n\omega_n}(S^k)$ is unbounded, where $\widetilde{j_0^n\omega_n}$ is any lift of $j_0^n\omega_n$ to the universal cover $\mathbf{R}^1 \times Q$ of $S^1 \times Q$. This shows that ω is not representable by a map.

Let $\tilde{Y}_n$ denote the universal cover of Y_n. Note that $\widetilde{X_n \times Q} = \tilde{X}_n \times Q$ has a metric which agrees locally with the metric on $X_n \times Q$, is invariant under covering translations, and agrees with the usual metric on $\mathbf{R}^1 \times B^{k+1} \times Q$. In particular, if f: $A \to X_n$ is a map, then $\operatorname{diam} \tilde{f}(A)$ is independent of the choice of the lifting $\tilde{f}$: $A \to \tilde{X}_n$ of f, provided A is path connected.

LEMMA. *If the map f: $X_n \times Q \to X_0 \times Q$ is such that*

$$f_*: \pi_1(X_n \times Q) \to \pi_1(X_0 \times Q)$$

is an isomorphism, then for each $N > 0$ there exists an $M > 0$ such that if $A \subset \tilde{X}_n \times Q$ is a subset with $\operatorname{diam} A > M$ *and $\tilde{f}$: $\tilde{X}_n \times Q \to \tilde{X}_0 \times Q$ covers f, then* $\operatorname{diam} \tilde{f}(A) > N$.

PROOF. This follows immediately from the fact that f is proper and from our choice of metrics.

We will now construct the desired shape morphism ω: $S^k \to Y$. It will be determined by an infinite word of the form $\omega = \sum_{i=1}^{\infty}(t^{k_i} + t^{-k_i})a_i$. Each such word defines a shape morphism ω: $S^k \to Y$ such that $j_n\omega$ is represented by $[\omega_n] = \sum_{i=1}^{n}(t^{k_i} + t^{-k_i})a_i$. Choose M_n as in the lemma above so that if $\operatorname{diam} A > M_n$, then

$\operatorname{diam} \widetilde{j_0^n}(A) > n + 1$. Pick k_n sufficiently large to guarantee that if α_n: $S^k \to X_n \times Q$ represents $[\omega_n]$, then $\tilde{\alpha}_n$: $S^k \to \tilde{X}_n \times Q$ has diam $\tilde{\alpha}_n(S^k) > M_n$. This can be done since diam $\tilde{\alpha}_n(S^k) \geq 2k_n$. Thus, $\operatorname{diam} \widetilde{j_0^n}\tilde{\alpha}_n(S^k) > n + 1$. Suppose there is a map α: $S^k \to Y$ representing the constructed shape morphism ω. We can set $\alpha_n = j_n\alpha$ and conclude that

$$\operatorname{diam} \widetilde{j_0\alpha}(S^k) = \operatorname{diam} \widetilde{j_0^n j_n \alpha}(S^k) = \operatorname{diam} \widetilde{j_0^n \alpha_n}(S^k) > n + 1,$$

for each n. This contradicts the compactness of S^k and completes the proof.

Appendix.

LEMMA. *Let $A = \varprojlim A_n$ with bonds p_n: $A_n \to A_{n-1}$ and with each A_n a compact ANR. If X is a compactum shape equivalent to A then there is a sequence of embeddings j_{n+1}: $A_{n+1} \times Q \to A_n \times Q$ such that $X = \varprojlim(A_n \times Q, j_n) = \bigcap_{n=1}^{\infty} j_1^n(A_n \times Q)$. Moreover, j_n is homotopic to $p_n \times \mathrm{id}_Q$.*

PROOF. If j: $A \times Q \to Q$ is a Z-embedding, then $j(A \times Q) = \bigcap_{i=1}^{\infty} M_i$, where each M_i is a Q-manifold neighborhood of $A \times Q$ homeomorphic to $A_i \times Q$ in such a way that the diagrams

$$\begin{array}{ccc} M_{i+1} & \to & M_i \\ \downarrow \cong & & \downarrow \cong \\ A_{i+1} \times Q & \xrightarrow{p_{i+1} \times \mathrm{id}_Q} & A_i \times Q \end{array}$$

commute up to homotopy. This is well known. See [1], for example, for a proof.

Since $A \times Q$ and X are shape equivalent, the proof of Chapman's complement theorem [1] produces an isotopy f_t: $Q \twoheadrightarrow Q$, $0 \leq t < 1$, such that f_t and f_t^{-1} are supported on smaller and smaller neighborhoods of $A \times Q$ and X, respectively. If $\{t_i\}$ is a sequence of real numbers, $0 \leq t_i < 1$, converging rapidly to 1, $X = \bigcap_{i=1}^{\infty} f_{t_i}(M_i)$ and $f_{t_i}|M_{i+1}$ is ambient isotopic to $f_{t_{i+1}}|M_{i+1}$ in M_i. This shows not only that neighborhoods of A are homeomorphic (simple homotopy equivalent) to neighborhoods of X but also that the homeomorphisms can be chosen coherently.

REFERENCES

1. T. A. Chapman, *Lectures on Hilbert cube manifolds*, CBMS Regional Conf. Ser. in Math., no. 28, Amer. Math. Soc., Providence, R.I., 1976.

2. S. Ferry, *A stable converse to the Vietoris-Smale Theorem with applications to shape theory*, Trans. Amer. Math. Soc. **261** (1980), 369–386.

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