CONTINUITY OF BEST APPROXIMANTS

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Abstract. Let $C_n$, $n \in \mathbb{N}$, be $\Phi$-closed lattices in an Orlicz-space $L_\Phi(\Omega, \mathcal{A}, \mu)$ and assume that $C_n$ increases or decreases to a $\Phi$-closed lattice $C_\infty$. Let $f_n$, $n \in \mathbb{N}$, be $\Phi$-measurable real valued functions with $f_n \to f$ $\mu$-a.e. and $\sup |f_n| \in L_\Phi$. If $g_n$ is a best $\Phi$-approximant of $f_n$ in $C_n$, it is shown that $\lim_{n \to \infty} g_n$ and $\lim_{n \to \infty} g_n$ are best $\Phi$-approximants of $f$ in $C_\infty$.

1. Introduction and notations. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function with $\Phi(0) = 0$ and $\Phi \not\equiv 0$. Denote by $L_\Phi(\Omega, \mathcal{A}, \mu)$ respectively $L_\infty(\Omega, \mathcal{A}, \mu)$ the system of all $\mu$-equivalence classes of $\mathcal{A}$-measurable functions $f$ such that $\int \Phi(\alpha|f|) \, d\mu < \infty$ for some $\alpha > 0$ respectively for all $\alpha > 0$. $L_\phi$ and $L_\infty$ are linear spaces with $L_\Phi \subseteq L_\infty$; if $\Phi(x) = x^p$ then $L_\Phi = L^p$ and we obtain the spaces $L_p$, $p > 1$. If $C \subseteq L_\Phi$ and $f \in L_\Phi$ denote by $\mu_\Phi(f|C)$ the system of all $g \in C$ fulfilling

$$\int \Phi(|f - g|) \, d\mu = \inf_{h \in C} \int \Phi(|f - h|) \, d\mu.$$ 

The elements of $\mu_\Phi(f|C)$ are called best $\Phi$-approximants of $f$, given $C$. The concept of best $\Phi$-approximants, given $C$, covers and unifies many important concepts of probability theory, e.g. the concepts in [1], [2], [6]; for more details see [4]. It is known that $\mu_\Phi(f|C) \neq \emptyset$ if $C$ is a lattice (i.e. $f, g \in C$ implies $f \land g, f \lor g \in C$) which is $\Phi$-closed (i.e. $f_n \in C, f \in L_\Phi$ and $f_n \uparrow f$ or $f_n \downarrow f$ imply $f \in C$); see Theorem 4 of [4]. In general, $\mu_\Phi(f|C)$ contains a lot of different elements; for instance if $\Phi(x) = x$ or if $C$ is not convex. This creates problems for proving limit theorems for best $\Phi$-approximants. In special cases—i.e. for $\Phi(x) = x^p$, $p > 1$, and special types of $C$—limit results for best $\Phi$-approximants of $f$, given $C$, are easier to obtain for varying $f$ than for varying $C$; but in all these cases best approximants are unique. In the general context, however, the case of varying $f$ is more complex. There exist limit theorems for best $\Phi$-approximants of martingale type (see Theorem 21 and Theorem 22 of [4])—i.e. limit theorems for $g_n \in \mu_\Phi(f|C_n)$ with varying $C_n$—but there exist no continuity theorems for best $\Phi$-approximants—i.e. limit theorems for $g_n \in \mu_\Phi(f_n|C)$ with varying $f_n$. It is the aim of this paper to close this gap. We prove a limit theorem for best $\Phi$-approximants $g_n \in \mu_\Phi(f_n|C_n)$ where as well the functions $f_n$ as the $\Phi$-closed lattices $C_n$ may vary with $n \in \mathbb{N}$. We apply this result to obtain continuity of best approximants in the Orlicz-space norm of $L_\Phi$. 

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2. The results. Throughout the following let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function with $\Phi(0) = 0$ and $\Phi \equiv 0$. Then $\Phi$ is a continuous function with $\lim_{t \to \infty} \Phi(t) = \infty$. If $C_n \subseteq L_\Phi$, $n \in \mathbb{N} \cup \{\infty\}$, we write $C_n \downarrow C_\infty$ if $C_n \subseteq C_{n+1}$, $n \in \mathbb{N}$, and $C_\infty = \bigcap_{n \in \mathbb{N}} C_n$. If $C_n$ are $\Phi$-closed lattices we write $C_n \uparrow C_\infty$ if $C_n \subseteq C_{n+1}$ and $C_\infty$ is the smallest $\Phi$-closed set containing $\bigcup_{n \in \mathbb{N}} C_n$; then $C_\infty$ is a lattice, too (see [4, p. 229]).

1. Theorem. Assume that $L_\Phi = L_\Phi^\infty$. Let $C_n \subseteq L_\Phi$, $n \in \mathbb{N}$, be $\Phi$-closed lattices with $C_n \downarrow C_\infty$ or $C_n \uparrow C_\infty$ and $f_n \in L_\Phi$, $n \in \mathbb{N}$, with $f_n \to f \mu$-a.e. and $\sup_{n \in \mathbb{N}} |f_n| \in L_\Phi$. Then for all $g_n \in \mu_\Phi(f_n|C_n)$, $n \in \mathbb{N}$,

(i) $\lim_{n \to \infty} g_n \in \mu_\Phi(f|C_\infty)$,
(ii) $\overline{\lim}_{n \to \infty} g_n \in \mu_\Phi(f|C_\infty)$.

Proof. Let $C_n \uparrow C_\infty$. We prove that for each $g \in \mu_\Phi(f|C_\infty)$

(1) $g \land \lim_{n \to \infty} g_n \in \mu_\Phi(f|C_\infty)$, $g \lor \overline{\lim}_{n \to \infty} g_n \in \mu_\Phi(f|C_\infty)$.

As

$$\lim_{n \to \infty} g_n = \left( g \lor \overline{\lim}_{n \to \infty} g_n \right) \land \lim_{n \to \infty} g_n$$

and

$$\overline{\lim}_{n \to \infty} g_n = \left( g \land \lim_{n \to \infty} g_n \right) \lor \overline{\lim}_{n \to \infty} g_n,$$

(1) implies (i).

Applying Lemma 3 to $C_k \supseteq \cdots \supseteq C_n \supseteq C_\infty$ we obtain for $n > k$

$$\int \Phi(|f_k \land \cdots \land f_n \land f - g_k \land \cdots \land g_n \land g|) \, d\mu$$

(2) $< \int \Phi(|f_k \land \cdots \land f_n \land f - g|) \, d\mu$

and

$$\int \Phi(|f_k \lor \cdots \lor f_n \lor f - g_k \lor \cdots \lor g_n \lor g|) \, d\mu$$

(3) $< \int \Phi(|f_k \lor \cdots \lor f_n \lor f - g|) \, d\mu$.

From (2) we obtain for each $k$ with $n \to \infty$ according to the Lemma of Fatou that

$$\int \Phi\left(|f \land \bigwedge_{n \geq k} f_n - g \land \bigwedge_{n \geq k} g_n|\right) \, d\mu$$

(4) $< \lim_{n \to \infty} \int \Phi(|f_k \land \cdots \land f_n \land f - g|) \, d\mu.$

Since $\sup_{n \in \mathbb{N}} |f_n| \in L_\Phi$ by assumption, (4) implies by the Theorem of Lebesgue that for each $k \in \mathbb{N}$

$$\int \Phi\left(|f \land \bigwedge_{n \geq k} f_n - g \land \bigwedge_{n \geq k} g_n|\right) \, d\mu < \int \Phi\left(|f \land \bigwedge_{n \geq k} f_n - g|\right) \, d\mu < \infty.$$
As \( f_n \to f \), \( \sup_{n \in \mathbb{N}} |f_n| \in L_\Phi \) we obtain from (5) with \( k \to \infty \) using on the left side the Lemma of Fatou and on the right side the Theorem of Lebesgue that

\[
\int \Phi\left( |f - g \wedge \lim_{n \in \mathbb{N}} g_n| \right) \, d\mu < \int \Phi(|f - g|) \, d\mu < \infty.
\]

From (5) we obtain that \( g \wedge \bigwedge_{n \geq k} g_n \in L_\Phi \). As \( g \wedge \bigwedge_{n \geq k} g_n < \bigwedge_{n \geq k} g_n < g_k \) this implies

\[
\bigwedge_{n \geq k} g_n \in L_\Phi.
\]

In the same way as (6) and (7) we obtain

\[
\int \Phi\left( |f - g \vee \lim_{n \in \mathbb{N}} g_n| \right) \, d\mu < \int \Phi(|f - g|) \, d\mu < \infty
\]

and

\[
\bigvee_{n \geq k} g_n \in L_\Phi.
\]

From (7) and (7)* applied to \( k = 1 \) we obtain (ii). As \( C_n \) are \( \Phi \)-closed lattices we obtain with (7) that \( \bigwedge_{n \geq j} g_n \in C_k \) for \( j > k \). As \( \lim_{n \in \mathbb{N}} g_n \in L_\Phi \) by (ii), this implies \( \lim_{n \in \mathbb{N}} g_n \in C_k \) for each \( k \in \mathbb{N} \) and hence \( \lim_{n \in \mathbb{N}} g_n \in C_\infty \). Similarly \( \lim_{n \in \mathbb{N}} g_n \in C_\infty \). Now (6), (6)* and \( g \in C_\infty \) imply (1). This finishes the proof for the decreasing case.

Now let \( C_n \uparrow C_\infty \). Applying Lemma 3 with \( C_n \supset C_{n-1} \supset \cdots \supset C_k \), \( k < n \), we obtain for \( k < n \)

\[
\int \Phi(|f_k \wedge \cdots \wedge f_n - g_k \wedge \cdots \wedge g_n|) \, d\mu < \int \Phi(|f_k \wedge \cdots \wedge f_n - g_k|) \, d\mu.
\]

Proceeding now as in the decreasing case, i.e. letting at first \( n \to \infty \) and then \( k \to \infty \) and using on the left sides the Lemma of Fatou and on the right sides the Theorem of Lebesgue we obtain

\[
\bigwedge_{n \geq k} g_n \in L_\Phi, \quad k \in \mathbb{N},
\]

and

\[
\int \Phi\left( |f - \lim_{n \in \mathbb{N}} g_n| \right) \, d\mu < \lim_{k \in \mathbb{N}} \int \Phi\left( \bigwedge_{n \geq k} f_n - g_k \right) \, d\mu.
\]

In the same way we obtain

\[
\bigvee_{n \geq k} g_n \in L_\Phi, \quad k \in \mathbb{N},
\]

and

\[
\int \Phi\left( |f - \lim_{n \in \mathbb{N}} g_n| \right) \, d\mu < \lim_{k \in \mathbb{N}} \int \Phi\left( \bigvee_{n \geq k} f_n - g_k \right) \, d\mu < \infty.
\]

Relations (8) and (8*) directly imply

\[
\sup_{n \in \mathbb{N}} |g_n| \in L_\Phi \quad \text{and} \quad \lim_{n \in \mathbb{N}} g_n, \quad \lim_{n \in \mathbb{N}} g_n \in C_\infty.
\]
Now apply Lemma 4 to \( h_k = |f_k - g_k| \) and \( r_k := |\cap_{n > k} f_n - f_k| \). Since \( \sup_{k \in \mathbb{N}} g_k \in L_\Phi \) by (10), \( \sup_{k \in \mathbb{N}} |f_k| \in L_\Phi \) and \( f_k \rightarrow f \) \( \mu \)-a.e. by assumption we have \( \sup_{k \in \mathbb{N}} h_k, \sup_{k \in \mathbb{N}} r_k \in L_\Phi \) and \( r_k \rightarrow 0 \) \( \mu \)-a.e., i.e. the assumptions of Lemma 4 are fulfilled. Hence we obtain

\[
\lim_{k \in \mathbb{N}} \int \Phi(h_k + r_k) \, d\mu = \lim_{k \in \mathbb{N}} \int \Phi(h_k) \, d\mu.
\]

Since

\[
\Phi \left( \left| \cap_{n > k} f_n - g_k \right| \right) < \Phi \left( \left| f_k - g_k \right| + \left| \cap_{n > k} f_n - f_k \right| \right) = \Phi(h_k + r_k),
\]

(9) and (11) imply

\[
\int \Phi \left( \left| f - \lim_{n \in \mathbb{N}} g_n \right| \right) \, d\mu < \lim_{k \in \mathbb{N}} \int \Phi \left( |f_k - g_k| \right) \, d\mu.
\]

According to (12) and (10) we get \( \lim_{n \in \mathbb{N}} g_n \in \mu_\Phi(f|C_\infty) \) if we show that for all \( g \in C_\infty \)

\[
\lim_{k \in \mathbb{N}} \int \Phi(|f_k - g_k|) \, d\mu < \int \Phi(|f - g|) \, d\mu.
\]

Let \( \hat{C} \) be the set of all \( g \in L_\Phi \) fulfilling (13). Since \( g_k \in \mu_\Phi(f_k|C_k), \Delta_k \uparrow, f_k \rightarrow f \) \( \mu \)-a.e. and \( \sup_{n \in \mathbb{N}} |f_n| \in L_\Phi \) it is easy to see that \( \hat{C} \) is \( \Phi \)-closed with \( \cup_{n \in \mathbb{N}} C_n \subset \hat{C} \). Hence \( C_\infty \subset \hat{C} \), i.e. (13) holds for all \( g \in \hat{C} \). Thus \( \lim_{n \in \mathbb{N}} g_n \in \mu_\Phi(f|C_\infty) \) is shown; the proof for \( \lim_{n \in \mathbb{N}} g_n \in \mu_\Phi(f|C_\infty) \) runs similarly (by using (9*) instead of (9)).

The martingale results, given in [4], hold for more general functions \( \Phi \) than convex functions, namely for so-called \( \mu \)-functions. We do not know whether also the preceding theorem is true for this more general concept; the proof of Theorem 1 heavily uses the convexity of \( \Phi \). Approximating \( \lim_{n \in \mathbb{N}} g_n \) \( \mu \)-a.e. by \( g_{\tau_n} \) where \( \tau_n \) is a sequence of finite stopping times for \( g_n, n \in \mathbb{N} \), it can be seen that Theorem 1 is true for \( \mu \)-functions in the special case that \( C_n \) is the system of \( \mathcal{B}_\tau \)-measurable functions in \( L_\Phi \), where \( \mathcal{B}_\tau \subset \mathcal{B} \) are \( \sigma \)-fields, and \( \mathcal{B}_\tau \uparrow \mathcal{B}_\infty \) or \( \mathcal{B}_\tau \downarrow \mathcal{B}_\infty \). However, this procedure fails for arbitrary \( \Phi \)-closed lattices \( C_n \).

If \( f \in L_\Phi \) put \( \|f\|_\Phi := \inf\{a > 0: \int \Phi(|f|/a) \, d\mu < 1\} \). Then \( \| \|_\Phi \) is a norm on \( L_\Phi \) and the spaces \( (L_\Phi, \| \|_\Phi) \) are Banach-spaces; the well-known Orlicz spaces (see [5, p. 46]). If \( C \subset L_\Phi \) and \( f \in L_\Phi \) we denote by \( \mu_{\Phi} \|f|C\) the set of all best \( \| \|_\Phi \)-approximants of \( f \), given \( C \), i.e. the set of all elements \( g \in C \) with

\[
\|f - g\|_\Phi = \inf\{\|f - h\|_\Phi: h \in C\}.
\]

The concept of best \( \| \|_\Phi \)-approximants and its connection with the concept of best \( \Phi \)-approximants has been investigated in [4]. If \( \Phi \) is strictly convex and if \( L_\Phi = L_\infty \), then for each \( \Phi \)-closed convex lattice \( C \subset L_\Phi \) and each \( f \in L_\Phi \) there exist a unique best \( \Phi \)-approximant and a unique best \( \| \|_\Phi \)-approximant of \( f \), given \( C \) (see Corollary 5 and Corollary 13 of [4]), we denote these unique elements by \( \mu_{\Phi}(f|C) \).
and \( \mu_\|_{\|} (f|C) \), respectively. Hence \( L_\Phi \ni f \to \mu_\Phi (f|C) \in L_\Phi \) and \( L_\Phi \ni f \to \mu_\|_{\|} (f|C) \in L_\Phi \) are operators on \( L_\Phi \) and the following result states the \( \| \|_\Phi \)-continuity of these operators.

2. Corollary. Let \( \Phi \) be strictly convex and assume that \( L_\Phi = L_\Phi^\infty \). Let \( C \subset L_\Phi \) be a \( \Phi \)-closed convex lattice and a cone. Then \( \mu_\Phi (\cdot|C) \) and \( \mu_\|_{\|} (\cdot|C) \) are \( \| \|_\Phi \)-continuous operators on \( L_\Phi \).

Proof. As \( L_\Phi = L_\Phi^\infty \) let us at first remark that

\[
\| h_n \|_\Phi \to 0 \text{ iff } \int \Phi(a|h_n|) \, d\mu \to 0 \text{ for all } a > 0.
\]

Let now \( \| f_n - f_0 \|_\Phi \to n\in\mathbb{N} 0 \) and \( N_1 \subset \mathbb{N} \) be a subsequence. It suffices to prove that there exists a subsequence \( N_2 \subset N_1 \) such that

\[
(1) \quad \| \mu_\Phi (f_n|C) - \mu_\Phi (f_0|C) \|_\Phi \to 0, \quad n \in N_2.
\]

\[
(2) \quad \| \mu_\|_{\|} (f_n|C) - \mu_\|_{\|} (f_0|C) \|_\Phi \to 0.
\]

Since \( \| f_n - f_0 \|_\Phi \to n\in\mathbb{N} 0 \) there exists a subsequence \( N_2 \subset N_1 \) such that

\[
(3) \quad f_n \to f_0 \text{ \mu-a.e.}
\]

and

\[
(4) \quad \sum_{n \in N_2} \| f_n - f_0 \|_\Phi < \infty.
\]

From (5) and \( L_\Phi = L_\Phi^\infty \) we obtain

\[
(5) \quad \sup_{n \in N_2} | f_n | < | f_0 | + \sum_{n \in N_2} | f_n - f_0 | \in L_\Phi.
\]

Now (4), (6) and Theorem 1 imply

\[
(6) \quad \mu_\Phi (f_n|C) \to \mu_\Phi (f_0|C) \text{ \mu-a.e.; } \sup_{n \in N_2} | \mu_\Phi (f_n|C) | \in L_\Phi.
\]

Using (1), \( L_\Phi = L_\Phi^\infty \) and the Theorem of Lebesgue, (7) implies (2). It remains to prove (3). Since \( \| f_n - f_0 \|_\Phi \to 0 \) and \( C \) is \( \| \|_\Phi \)-closed (see Theorem 10 of [4]) it is easy to see that

\[
(8) \quad \delta_n := \| f_n - \mu_\|_{\|} (f_n|C) \|_\Phi \to n\in\mathbb{N} \| f_0 - \mu_\|_{\|} (f_0|C) \|_\Phi =: \delta_0.
\]

Let w.l.g. \( \delta_0 > 0 \); hence w.l.g. \( \delta_n > 0 \) for all \( n \in \mathbb{N} \). According to Corollary 8 of [4] we have, as \( C \) is a cone, that

\[
(9) \quad \mu_\|_{\|} (f_n|C) \mu_\Phi \left( \frac{1}{\delta_n} \right) f_n|C), \quad n \in \mathbb{N} \cup \{0\}.
\]

Since \( f_n \to f_0 \) and \( \delta_n \to \delta_0 \) by (8), we have

\[
\frac{1}{\delta_n} \to f_n \to \delta_0 f_0.
\]
Hence the continuity of $\mu_\Phi(\cdot | C)$ implies
\[
\left\| \mu_\Phi\left( \frac{1}{\delta_n} f_n | C \right) - \mu_\Phi\left( \frac{1}{\delta_0} f_0 | C \right) \right\|_{\Phi^n} \rightarrow 0.
\]
Together with (9) and (8) this yields (3).

For the special case that $C$ is the system of measurable functions with respect to a $\sigma$-field the assertion of Corollary 2 follows from Satz 5.10 of [3]. The methods used there are closely related to this special type of $C$ and cannot be transferred to arbitrary $\Phi$-closed convex lattices.

The following lemmas are the main tools for the proof of Theorem 1.

3. Lemma. Assume that $L_\Phi = L_\Phi^\infty$. Let $C_i \subset L_\Phi$, $i = 1, \ldots, n$, be $\Phi$-closed lattices with $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n$. If $f_i \in L_\Phi$ and $g_i \in \mu_\Phi(f_i | C_i)$, $i = 1, \ldots, n$ then

(i) $\int \Phi(\{f_i \wedge \cdots \wedge f_n - g_i \wedge \cdots \wedge g_n\}) \, d\mu < \int \Phi(\{f_i \wedge \cdots \wedge f_n - g_{i+1} \wedge \cdots \wedge g_n\}) \, d\mu$,

(ii) $\int \Phi(\{f_i \vee \cdots \vee f_n - g_i \vee \cdots \vee g_n\}) \, d\mu < \int \Phi(\{f_i \vee \cdots \vee f_n - g_{i+1} \vee \cdots \vee g_n\}) \, d\mu$.

Proof. To show (i) it suffices to prove that for $j < n$
\[
\int \Phi(\{f_1 \wedge \cdots \wedge f_n - g_j \wedge \cdots \wedge g_n\}) \, d\mu < \int \Phi(\{f_1 \wedge \cdots \wedge f_n - g_{j+1} \wedge \cdots \wedge g_n\}) \, d\mu.
\]
As $\Phi$ is convex, Lemma 20 of [4] implies
\[
\Phi(\{f_1 \wedge \cdots \wedge f_n - g_j \wedge (g_{j+1} \wedge \cdots \wedge g_n)\})
\leq \Phi(\{f_j - g_j \wedge (g_{j+1} \wedge \cdots \wedge g_n)\}) + \Phi(\{f_j - g_j\}).
\]
Since $C_j$ is a lattice and $g_i \in C_i \subset C_j$ for $i > j$ we have $g_j \vee (g_{j+1} \wedge \cdots \wedge g_n) \in C_j$. As $g_j \in \mu_\Phi(f_j | C_j)$ we obtain
\[
\int \Phi(\{f_j - g_j\}) \, d\mu < \int \Phi(\{f_j - g_j \vee (g_{j+1} \wedge \cdots \wedge g_n)\}) \, d\mu.
\]
Using (3) integration of (2) yields (1). This proves (i); the proof for (ii) runs by interchanging $\vee$ and $\wedge$.

4. Lemma. Assume that $L_\Phi = L_\Phi^\infty$. Let $0 < h_k$, $r_k \in L_\Phi$ and assume that $\sup_{k \in \mathbb{N}} h_k$, $\sup_{k \in \mathbb{N}} r_k \in L_\Phi$ and $r_k \to 0 \mu$-a.e. Then
\[
\int \Phi(h_k + r_k) \, d\mu - \int \Phi(h_k) \, d\mu \rightarrow 0.
\]
Proof. Let $\Phi'_+$ be the right derivative of $\Phi$. Then $\Phi'_+$ is nondecreasing and $\Phi(x) = \int_0^x \Phi'_+(t) \, dt$ (see e.g. [5]). Hence for all $k \in \mathbb{N}$
\[
(*) \quad \Phi(h_k + r_k) - \Phi(h_k) = \int_{h_k}^{h_k + r_k} \Phi'_+(t) \, dt < r_k \Phi'_+(h_k + r_k) < r \Phi'_+(h + r)
\]
with $r := \sup_{k \in \mathbb{N}} r_k \in L_\Phi$ and $h := \sup_{k \in \mathbb{N}} h_k \in L_\Phi$. 

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By the Theorem of Lebesgue (*) directly implies the assertion if we show $r\Phi'_+(h + r) \in L_1$. As $0 \leq x\Phi'_+(x) < \int_0^x \Phi'_+(t) \, dt \leq \Phi(2x)$ and $L_\Phi = L_\Phi^\infty$, we have $g\Phi'_+(g) \in L_1$ if $0 \leq g \in L_\Phi$. Applying this to $g = h + r \in L_\Phi$ we obtain $r\Phi'_+(h + r) \in L_1$.

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