CONVERGENCE OF BEST BEST $L_{\infty}$-APPROXIMATIONS

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ABSTRACT. Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $(\mathcal{B}_i)_{i=1}^{\infty}$ be an increasing sequence of subsigma algebras of $\mathcal{A}$. Let $A = L_\infty(\Omega, \mathcal{A}, \mu)$, let $B_i = L_\infty(\Omega, \mathcal{B}_i, \mu)$, $i > 1$, and let $f \in A$. Let $f_i$ denote the best best $L_\infty$-approximation to $f$ by elements of $B_i$. It is shown that $\lim_{i} f_i(x)$ exists a.e.

We begin with a brief introduction in which we present notation and terminology, a related result [1] for $1 < p < \infty$, and some results from [2] that will be used to establish a.e. convergence of the sequence $\{f_i\}$.

Let $\mathcal{B}_\infty$ denote the subsigma algebra generated by the algebra $\bigcup_i \mathcal{B}_i$. For $1 < p < \infty$, $1 < i < \infty$ and $g \in L_p(\Omega, \mathcal{A}, \mu)$, let $g_{i,p}$ denote the best $L_p$-approximation to $g$ by elements of $L_p(\Omega, \mathcal{B}_i, \mu)$; T. Ando and I. Amemiya [1] showed that $\lim_{i} g_{i,p} = g$ a.e. and in $L_p$. It is shown in [2] that if $g \in A$, then for $1 < i < \infty$, $\lim_{i} g_{i,p}$ exists a.e. and is the best $L_\infty$-approximation, $g_{i,\infty}$, to $g$ by $\mathcal{B}_i$-measurable functions. To simplify the notation, we let $f$ be a fixed element of $A$ and, without loss of generality, we suppose that $0 < f < 1$; furthermore, we denote $f_{i,\infty}$ by $f_i$. Our proof of the fact that the sequence $\{f_i\}$ converges a.e. uses some technical results from [2] that we introduce next. To simplify the notation during this introduction, suppress $i$ from $B_i$: $\mathcal{B}$ is a subsigma algebra of $\mathcal{A}$ and $B = L_\infty(\Omega, \mathcal{B}, \mu)$.

Let $\mathcal{P}$ denote the set of denumerable partitions of $\Omega$ by elements of $\mathcal{B}$.

For $E \in \mathcal{A}$, let $O(E)$ denote the essential oscillation of $f$ on $E$: $O(E) = O(f, E) = \text{essup}(f, E) - \text{essinf}(f, E)$, where $\text{essup} (f, E) = \text{essinf}(f, E) = 0$ if $\mu(E) = 0$ and for $\mu(E) > 0$

\[ u(E) = \text{essup}(f, E) = \inf\{\lambda; \mu(\{x \in E; f(x) > \lambda\}) = 0\} \]

and

\[ l(E) = \text{essinf}(f, E) = \sup\{\lambda; \mu(\{x \in E; f(x) < \lambda\}) = 0\}. \]

Let $d(g, B)$ denote the distance from an element $g$ of $A$ to the subspace $B$ of $A$.

Next we recall two lemmas from [2]. Lemma 1 shows that $\mathcal{P}$ can be used to estimate $d(f, B)$. Lemma 2 asserts that the flexibility afforded by $\mathcal{P}$ permits us to replace an inf by a min. The partitions corresponding to each min provide an equivalence class of elements of $\mathcal{B}$. These equivalence classes comprise a monotone family parameterized by the positive reals.

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Lemma 1. \( d = d(f, B) = (1/2) \inf_{E \in \mathcal{P}} \sup \{ O(E); E \in \pi, \mu(E) > 0 \} \).

Lemma 2. For \( h > 0 \) and \( \pi \in \mathcal{P} \), let \( \delta(h, \pi) = \{ \Sigma \mu(E); E \in \pi, O(E) > h \} \) and let \( \delta_h = \inf \{ \delta(h, \pi); \pi \in \mathcal{P} \} \). Then there exists \( \pi \) such that \( \delta_h = \delta(h, \pi) = \mu(E^h) \), where \( E^h = \bigcup \{ E; E \in \pi, O(E) > h \} \).

Lemmas 1 and 2 assure (i) \( \delta_h = 0 \) if \( h > 2d \) and (ii) if \( h < 2d \), then there exists \( \pi \) such that \( \delta(h, \pi) = \delta_h > 0 \). Notice also that if \( \delta(h, \pi) = \delta_h, E \in \mathcal{P}, E \in E^h \) and \( \mu(E) > 0 \), then \( O(E) > h \); thus, \( E^h \) is uniquely determined up to a set of measure zero by the equation \( \delta(h, \pi) = \delta_h \), so we can denote it by \( E_h \).

Now observe that if \( h_1 < h_2 \), then \( \mu(E_{h_1} - E_{h_2}) = 0 \).

Let \( m(E) = (1/2)(l(E) + u(E)), E \in \mathcal{P}, \mu(E) > 0 \).

The proof of the fourth lemma in [2] implies a lemma that we record for future use.

Lemma 3. Let \( h > 2\gamma > 0 \), let \( E(h, \gamma) = E_h - E_{h+\gamma} = \{ \bigcup_j F_j; F_j \in \mathcal{P}, 0 < \mu(F_j), h < O(F_j) < h + \gamma \} \), and let \( \hat{f} \) denote the best best \( L_\infty \) approximation to \( f \) by elements of \( B \). Then \( \mu(\{ x \in F_j; |\hat{f}(x) - m(F_j)| > \gamma/2 \}) = 0 \); for almost all points \( x \in F_j, |\hat{f}(x) - m(F_j)| < \gamma/2 \).

Now we are ready to show that \( \{ f_i \} \) converges a.e. To this end, let \( 0 < \epsilon < 1 \). Let \( \gamma > 0 \) with \( 4\gamma < \epsilon \) and let \( M \) be the smallest positive integer for which \( \epsilon + M\gamma > 1 \). Reintroduce the index \( i \) to the notation (e.g. for \( \mathcal{P}_i, E_h \) becomes \( E_{ih} \)). Notice that \( E_{ih} \supset E_{j,h}, i < j, h > 0 \). For \( h > 0 \), let \( E_h = \bigcap_i E_{ih} \) and let \( D_{ih} = E_{ih} - E_h \). Let \( \eta > 0 \) satisfy the inequality \( 2\eta(M + 1) < \epsilon \), and let \( n \) be a positive integer such that \( \mu(D_{n+k\eta}) < \eta, k = 0, 1, \ldots, M \). Let \( D_n = \bigcup_{k=0}^M D_{n+k\eta} \), and let \( G_n = E_{nx} - D_n \). Then \( \{ \Omega - E_{nx}, G_n, D_n \} \) is a partition of \( \Omega \). Lemma 1 and Theorem 2 of [2] imply that \( |f_i - f| < \epsilon/2 \) on \( (\Omega - E_{nx}) \cup (\Omega - E_{n+k\eta}) \), \( i > n \). Notice that \( \mu(D_n) < \epsilon \). If we show that, for \( i > n, |f_i - f_n| < \epsilon/2 \) on \( G_n \), then we will have shown that for each \( \epsilon > 0 \) there is a set \( D = D_n \) and a positive integer \( n \) such that \( \mu(D) < \epsilon \) and \( |f_i(x) - f_i(x)| < \epsilon, i, j > n, x \notin D \); since this latter situation is equivalent to a.e. convergence of the sequence \( \{ f_i \} \) we will be done. So, let \( S_{i,k} = (E_{ix+k\eta} - E_{ix+(k+1)\eta}) \). For \( n < j < i, (S_{j,k} - S_{i,k}) \subset (E_{nx+k\eta} - E_{x+k\eta}) \) because \( E_{j,k} \supset E_{i,k}, h > 0 \). Now observe that

\[
E_{nx} = \bigcup_{k=0}^M S_{n,k} = G_n, \cup H_n, \ni.
\]

where \( G_n = \bigcup_{k=0}^M (S_{n,k} - S_{i,k}) \subset D_n \) and \( H_n = \bigcup_{k=0}^M (S_{n,k} \cap S_{i,k}) \subset G_n \). Since \( H_n \supset G_n \), it suffices to show that \( |f_n - f_i| < \epsilon/2 \) on \( H_n, i > n \), as follows. Fix \( k \) to simplify the notation and let \( h = \epsilon + k\gamma \). Then

\[
S_{n,k} = \bigcup_l \{ F_{n,l}; \mu(F_{n,l}) > 0, F_{n,l} \in \mathcal{P}_n, h < O(F_{n,l}) < h + \gamma \}.
\]

and

\[
S_{i,k} = \bigcup_m \{ F_{i,m}; \mu(F_{i,m}) > 0, F_{i,m} \in \mathcal{P}_m, h < O(F_{i,m}) < h + \gamma \}.
\]
So $S_{n,k} \cap S_{i,k} = \bigcup_{i,m}(F_{n,i} \cap F_{i,m})$. Fix $i$, $l$, $m$ and let $H$ denote $F_{n,i} \cap F_{i,m}$. Observe that $H \in \mathcal{F}_l$. Suppose $\mu(H) > 0$; then $h < O(H) < \max\{O(F_{n,i}), O(F_{i,m})\} < h + \gamma$ because $H \subseteq E_{i,h}$. A computation verifies that $|m(F_{n,i}) - m(F_{i,m})| < \gamma/2$. Finally we apply Lemma 3 and obtain $|f_n(x) - f_l(x)| < |f_n(x) - m(F_{n,i})| + |m(F_{n,i}) - m(F_{i,m})| + |m(F_{i,m}) - f_l(x)| < 2\gamma < \epsilon/2$ for $x \in H$. Thus, the following theorem is established.

**Theorem.** Let $f \in A$. Then the sequence $\{f_i\}$ of best best $L_\infty$-approximations to $f$ is bounded by $\|f\|_\infty$ and converges a.e.

We conclude with an example to illustrate the fact that $\{f_n\}$ need not converge to $f_\infty$.

**Example.** Let $\Omega = [0, 1)$, $\mathcal{B}$ be the Borel sets in $\Omega$, $\mu$ denote Lebesgue measure, and let $\mathcal{B}_n$ be generated by $\{(i-1)/2^n, i/2^n); 1 \leq i \leq 2^n\}$. Let $E$ be a countable union of closed subsets of $\Omega$ such that if $0 < u < v < 1$, then both $E$ and $\Omega - E$ intersect $(u, v)$ in a set of positive measure (cf. [3, p. 59]). Let $f$ be the indicator function $I_E$ of $E$ (i.e., $f(x) = 1$ if $x \in E$ and $f(x) = 0$ if $x \in \Omega - E$). Then $f = f_\infty$; but $f_n \equiv 1/2$, $n = 1, 2, \ldots$.

**References**


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