THE DISTRIBUTION FUNCTION IN THE MORREY SPACE

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Abstract. For $1 < p < \infty$, we consider $p$-integrable functions on a finite cube $Q_0$ in $\mathbb{R}^n$, satisfying

$$\left( \frac{1}{|Q|} \int_{Q} |f(x) - f_Q|^p \, dx \right)^{1/p} < C \varphi(|Q|)$$

for every parallel subcube $Q$ of $Q_0$, where $|Q|$ denotes the volume of $Q$, $f_Q$ is the mean value of $f$ over $Q$ and $\varphi(t)$ is a nonnegative function defined in $(0, \infty)$, such that $\varphi(t)$ is nonincreasing near zero, $\varphi(t) \to \infty$ as $t \to 0$, and $\varphi''(t)$ is nondecreasing near zero. The constant $C$ does not depend on $Q$. Let $g$ be a nonnegative $p$-integrable function $g: (0, 1) \to \mathbb{R}$ such that $g$ is nonincreasing and $g(t) \to \infty$ as $t \to 0$. We prove here that there exist a cube $Q_0$ and a function $f$ satisfying condition (1) for every parallel subcube $Q$ of $Q_0$, such that $\delta_f(\lambda) > C_1 \delta_g(\lambda)$ for $\lambda > \lambda_0$, $C_1 > 0$, where $\delta(\lambda)$ denotes the distribution function.

John and Nirenberg have introduced in [1] the functions of bounded mean oscillation. An integrable function $f$ on a finite cube $Q_0$ in $\mathbb{R}^n$ is called a BMO function if there exists a constant $C > 0$ such that

$$\frac{1}{|Q|} \int_{Q} |f(x) - f_Q| \, dx \leq C$$

for every parallel subcube $Q$ of $Q_0$, where $|Q|$ denotes the volume of $Q$ and $f_Q$ is the mean value of $f$ over $Q$.

These authors have shown that the distribution function of a function of bounded mean oscillation decreases exponentially. More exactly, there exist constants $C, \alpha > 0$ such that

$$\delta_{f-f_Q}(\lambda) = \text{meas}\{x \in Q_0 : |f(x) - f_Q| > \lambda\} \leq Ce^{-\alpha \lambda} |Q_0| \quad \text{for every } \lambda > 0.$$ 

This implies that the BMO functions satisfy additional conditions. Actually, they belong to $L^p(Q_0)$ for all $p < \infty$ and they satisfy

$$\left( \frac{1}{|Q|} \int_{Q} |f(x) - f_Q|^p \, dx \right)^{1/p} \leq C.$$
Now, we are led to consider more general spaces, for instance, those $p$-integrable functions $f$ on $Q_0$ satisfying

\[ \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p} < C|Q|^{-\alpha} \quad \text{for some } 0 < \alpha < 1/p. \]

It would be interesting to obtain some estimate for the distribution function in this space. However, it does not seem to be possible. Actually, we show this in a more general space.

In fact, let $\varphi(t)$ be a nonnegative function defined in $(0, \infty)$ such that $\varphi(t)$ does not increase near zero, $\varphi(t) \to \infty$ as $t \to 0$, and $t \varphi^p(t)$ does not decrease near zero. We say that a function $f$ that is $p$-integrable over a cube $Q_0$ in $\mathbb{R}^n$ belongs to the space $M^p_\varphi(Q_0)$ if it satisfies $\left( |Q|^{-1} \int_Q |f(x) - f_Q|^p \, dx \right)^{1/p} < C\varphi(|Q|)$ for every parallel subcube $Q$ of $Q_0$.

When $\varphi(t) = t^{-\alpha}$, $0 < \alpha < 1/p$, we get the Morrey space, that is the functions satisfying (2). Now, we prove the following result.

**Proposition.** Let $g: (0, 1) \to \mathbb{R}$ be a nonnegative, nonincreasing $p$-integrable function such that $g(t) \to \infty$ as $t \to 0$. Then, there exist a cube $Q_0$, a function $f \in M^p_\varphi(Q_0)$ and two constants $C, \lambda_0 > 0$ such that

\[ \delta_f(\lambda) > C_1 \delta_g(\lambda) \quad \text{for } \lambda > \lambda_0. \]

**Proof.** First, we prove the assertion in one variable. The general case will follow from this one.

According to the hypothesis, there exists $a < 1$ such that $\varphi(t)$ does not increase for $t < a$, $t \varphi^p(t)$ does not decrease there, and $\varphi(a) > 0$. We can also suppose that $\varphi(a) = 1$, so we get $\varphi(t) > 1$ for $t < a$. We suppose also that $\int_0^1 g(t)^p \, dt < a/4^p$. On the other hand, we complete the definition of the function $g$ in such a way that it remains left continuous.

Let $h > 2$ be the first natural number such that $g(t) < 2^{h-1}$, for some $t$. For each $k > h + 1$, we consider the interval $I_k = (x_k, x_{k-1})$, such that $2^{k-2} < g(t) < 2^{k-1}$ in $I_k$.

We assert that the sequence $\{x_k\}$ converges to zero. In fact, since it is decreasing, it has a limit $L > 0$. $L$ must be zero because, by construction, we have $g(x_k) > 2^{k-2}$, $k > h + 1$. Furthermore, the length of the intervals $I_k$ decreases as a geometric progression. In fact,

\[ |I_k|^{p(k-2)} < \int_{I_k} g(t)^p \, dt \leq \frac{a}{4^p}, \]

so that $|I_k| < a/2^{pk}$ for $k > h + 1$.

Now, we define a step function $m(t)$ as

\[ m(t) = 2^{k-1} \text{ in } I_k, \text{ for } k > h + 1. \]

Clearly, $\delta_m(\lambda) > \delta_g(\lambda)$ for $\lambda > 2^{h-1}$. Now, we will replace each interval $I_k$ by another one, $J_k = (y_k, y_{k-1})$, of length $|J_k| = 2^{pk}|I_k| = 2^p \int_{I_k} m(t)^p \, dt$. 

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We assert that \( \sum_{k \geq h+1} |J_k| < \infty \). In fact,

\[
\sum_{k \geq h+1} |J_k| = \sum_{k \geq h+1} 2^{nk} |I_k| = 4^p \sum_{k \geq h+1} 2^{p(k-2)} |I_k|
\]

\[
< 4^p \sum_{k \geq h+1} \int_{I_k} g(t)^p \, dt \leq 4^p \int_0^1 g(t)^p \, dt < a.
\]

Now, we will define a function \( f(t) \) on the interval \((0, y_h)\) in the following way. We fix one interval \( I_k \) and we divide it into \( n_k \) subintervals of length \( \delta_k = |I_k|/n_k \). The number \( n_k \) will be selected later. Now, let \( \epsilon_k = (|J_k| - |I_k|)/(n_k - 1) \). We divide the interval \( J_k \) into \( 2n_k - 1 \) subintervals of length \( \delta_k \) and \( \epsilon_k \) alternatively. We define \( f(t) \) as \( 2^{k-1} \) in the intervals of length \( \delta_k \) and zero in the others.

Over each interval \( J_k \) the measure of the set where \( f(t) \) does not vanish is exactly \( |I_k| \). Furthermore, \( f(t) \) coincides with \( m(t) \) on that set. So, both functions have the same distribution function.

Now, we assert that selecting the number \( n_k \) in each interval \( J_k \) in a correct way, we get \( f(t) \in M_p^\varphi((0, y_h)) \). Actually, we will prove that there exists a constant \( C > 0 \) such that

\[
\int_J f(t)^p \, dt \leq C |J| \varphi^p(|J|)
\]

for every subinterval \( J \) of \((0, y_h)\). This will clearly imply that \( f \in M_p^\varphi((0, y_h)) \).
First, let us consider the interval $J = (0, y_{L-1})$, for some $L$,
\[
\int f(t)^p \, dt = \sum_{k > L} \int f(t)^p \, dt = \sum_{k > L} 2^{p(k-1)}|I_k| = 2^{-p} \sum_{k > L} |J_k| = 2^{-p}|J|.
\]
As was shown above, $\sum_{k > L} |J_k| < a$; thus, $\varphi(|J|) > 1$ or $|J|^{p}(|J|) > |J|$. So, we get
\[
\int f(t)^p \, dt < 2^{-p}|J|^{p}(|J|).
\]
Now, let us consider an interval $J = (y_x, y_{L-1})$. In the same way, we obtain
\[
\int f(t)^p \, dt = \sum_{k} \int f(t)^p \, dt = \sum_{k} 2^{p(k-1)}|I_k| = 2^{-p} \sum_{k} |J_k| = 2^{-p}|J| < 2^{-p}|J|^{p}(|J|).
\]
Now, we will consider an interval $J$ contained in one of the intervals $J_k$. We will select the number $n_k$ in order to obtain the desired inequality over this interval.

Since we have supposed the index $k$ to be fixed, we will write simply $n$, $\delta$, $\epsilon$. We first assume that there are $j$ intervals of length $\delta$, which cover the interval $J$, in the following sense.

\[
\delta
\]

\[
\begin{array}{c}
1 \quad 2 \quad \ldots \quad j
\end{array}
\]

It would be desirable to obtain the inequality
\[
2^{p(k-1)}j\delta \leq (j\delta + (j - 1)\epsilon)\varphi^{p}(j\delta + (j - 1)\epsilon), \quad 1 \leq j \leq n.
\]

Since $\delta = |I_k|/n$, $\epsilon = (|J_k| - |I_k|)/n - 1 = (2^{p} - 1)|I_k|/n - 1$, we can write the above inequality in the form
\[
2^{p(k-1)}j \left|\frac{I_k}{n}\right| \leq \left(j \left|\frac{I_k}{n}\right| + (j - 1)(2^{p} - 1)\left|\frac{I_k}{n - 1}\right|\right)\varphi^{p}\left(j \left|\frac{I_k}{n}\right| + (j - 1)(2^{p} - 1)\left|\frac{I_k}{n - 1}\right|\right)
\]
or,
\[
2^{p(k-1)} \leq \left(1 + \frac{j - 1}{j \left|\frac{I_k}{n - 1}\right|} (2^{p} - 1)\right) \varphi^{p}\left(j \left|\frac{I_k}{n}\right| + (j - 1)(2^{p} - 1)\right).
\]
As we saw above, $|I_k| < a/2^{p}$. Moreover, $(j - 1)/(n - 1) < j/n$. Thus,
\[
|I_k| \left(j \left|\frac{I_k}{n}\right| + j \left|\frac{I_k}{n - 1}\right| (2^{p} - 1)\right) \leq \frac{a}{2^{p}} \left(j \left|\frac{I_k}{n}\right| + j \left|\frac{I_k}{n - 1}\right| (2^{p} - 1)\right) = a \frac{j}{n} < a.
\]
Since we have supposed that $\varphi$ is a nonincreasing function for $t < a$, we obtain

$$\varphi\left(|I_k|\left(\frac{j}{n} + \frac{j - 1}{n - 1}(2^{pk} - 1)\right)\right) > \varphi\left(a\frac{j}{n}\right) > \varphi(a) = 1.$$  

Thus, it suffices to find a natural number $n$ so that

$$2^{p(k-1)} \leq \left(1 + \frac{n}{j} \frac{j - 1}{n - 1}(2^{pk} - 1)\right)\varphi^p\left(a\frac{j}{n}\right), \quad 1 \leq j \leq n,$$

for $k > h + 1$ fixed. Since $\varphi(t) \to \infty$ as $t \to 0$, there exists $0 < r(k, p) < a$ such that $aj/n < r$ which implies $2^{p(k-1)} < \varphi^p(aj/n)$. Thus, when $j/n < r/a$, we get the desired inequality.

Now, we suppose $1 \leq n/j < a/r$, and we will select $n$ in such a way that

$$2^{p(k-1)} \leq 1 + \frac{n}{j} \frac{j - 1}{n - 1}(2^{pk} - 1).$$

Since $(j - 1)/j = 1 - 1/j$ increases as $j$ increases, the worst case occurs when $n/j = a/r$; that is,

$$2^{p(k-1)} \leq 1 + \frac{nr/a - 1}{n - 1}(2^{pk} - 1).$$

From this inequality, we deduce that selecting $n > (a/r - \theta)/(1 - \theta)$, where $\theta = (2^{p(k-1)} - 1)/(2^{pk} - 1)$, we obtain the desired inequality for the subinterval $J$. Now, we suppose that $J$ is contained in one of the intervals of length $\delta_k$, for $k$ fixed.

In this case, it suffices to satisfy the inequality

$$2^{p(k-1)}|J| \leq |J|\varphi^p(|J|),$$

or, $2^{k-1} \leq \varphi(|J|)$.

Since $|J| < \delta_k$, we will have $\varphi(|J|) > \varphi(\delta_k)$; so that, it suffices to obtain $2^{k-1} < \varphi(\delta_k)$. But this is the inequality above, for $j = 1$.

In the same way, we can prove the inequality for a given subinterval $J$ of some interval $J_k$. We merely have to use that the function $t\varphi^p(t)$ does not decrease for $t < a$.

Finally, let us consider a subinterval $J$ of the interval $(0, y_h)$. We can divide $J$ into at most three intervals. One of them is a union of some intervals $J_k$, and the others are contained in some other intervals $J_k$ and $J_{k-1}$. Thus, according to all we have said above, and using again the fact that $t\varphi^p(t)$ is a nondecreasing function, we obtain the inequality. This concludes the one variable case.

In the general case, we argue as follows. Let $f(t)$ be a function in the space $M^p_\varphi((0, y_h))$, satisfying the desired hypothesis. Let $Q_0 = \{(t_1, \ldots, t_n)|0 < t_j < y_h, j = 1, \ldots, n\}$. We define the function $F(t_1, \ldots, t_n)$ as

$$F(t_1, \ldots, t_n) = f(t_1).$$

We assert that $F \in M^p_\varphi(Q_0)$.

In fact, let $Q$ be a parallel subcube of $Q_0$; we can write $Q = S_1 \times \cdots \times S_n$, where $S_j$ are subintervals of the same length of $(0, y_h)$. Thus,

$$\int_Q F(t_1, \ldots, t_n)^p \, dt_1 \cdots dt_n = |S_2| \cdots |S_n| \int_{S_1} f(t_1)^p \, dt_1 < C|Q|\varphi^p(|S_1|).$$
Since \(|S_j| < y_h < a < 1\), we get \(|Q| = |S_1| \cdots |S_n| < |S_1| < a\). So that

\[ \varphi(|S_1|) < \varphi(|Q|) \quad \text{or} \quad \varphi^* (|S_1|) < \varphi^* (|Q|). \]

On the other hand, we have also that \(\delta_F (\lambda) = y_h^{-n - \delta} \) for \(\lambda > 0\). This completes the proof.

**Remark.** In [2], the definition of the Morrey space appears in a slightly different way. Working over cubes, that definition may be stated as follows.

A \(p\)-integrable function \(f\) on a finite cube \(Q_0\) in \(\mathbb{R}^n\) belongs to the Morrey space of order \(\alpha\), \(0 < \alpha < 1/p\), if

\[
\sup_{x \in \overline{Q}_0} \inf_{c \in C} \left[ |Q(x)|^{p \alpha - 1} \int_{Q(x) \cap Q_0} |f(y) - c|^p \, dy \right]^{1/p} < \infty
\]

where \(\overline{Q}_0\) means the closure of \(Q_0\) and, given \(x \in \overline{Q}_0\), \(Q(x)\) is a cube centered in \(x\), parallel to \(Q_0\).

Actually, Campanato has shown that it is the same to consider

\[
\sup_{x \in \overline{Q}_0} \left[ |Q(x)|^{p \alpha - 1} \int_{Q(x) \cap Q_0} |f(y)|^p \, dy \right]^{1/p} < \infty
\]

(see [3]). From this, we are led to consider those \(p\)-integrable functions on \(Q_0\) such that

\[
\int_{Q(x) \cap Q_0} |f(y)|^p \, dy \leq C |Q(x)|^{p \alpha} \varphi^p(|Q(x)|) \quad \text{for all} \quad x \in \overline{Q}_0, |Q(x)| < |Q_0|.
\]

The function \(F(t_1, \ldots, t_n)\) constructed in the above proposition satisfies this inequality, in fact let \(Q(x) \cap Q_0 = S_1 \times \cdots \times S_n\), where \(S_j\) are subintervals of \((0, y_h)\). Then

\[
\int_{Q(x) \cap Q_0} F(t_1, \ldots, t_n)^p \, dt_1 \cdots dt_n = |S_2| \cdots |S_n| \int_{S_1} f(t_1)^p \, dt_1 < C |Q(x) \cap Q_0| \varphi^p(|S_1|).
\]

Since \(|S_j| < y_h < a < 1\), we have \(a > |S_1| > |S_1| \cdots |S_n| = |Q(x) \cap Q_0|\). The function \(\varphi(t)\) is nonincreasing and the function \(t^p \varphi(t)\) is nondecreasing for \(t < a\), so we get

\[
|Q(x) \cap Q_0| \varphi^p(|S_1|) \leq |Q(x) \cap Q_0| \varphi^p(|Q(x) \cap Q_0|) < |Q(x)| \varphi^p(|Q(x)|).
\]

Actually, we have proved that

\[
\int_{Q(x) \cap Q_0} F(t_1, \ldots, t_n)^p \, dt_1 \cdots dt_n \leq C |Q(x) \cap Q_0|^{p \alpha} |Q(x)|^{p \alpha} \varphi^p(|Q(x)|^{p \alpha}).
\]

This concludes the remark.
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REFERENCES


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