**L₀ IS ω-TRANSITIVE**

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**ABSTRACT.** Let $L₀$ be the space of measurable functions on the unit interval. Let $F$ and $G$ be two subspaces of $L₀$, each isomorphic to the space of all sequences. It is proved that there is a linear homeomorphism of $L₀$ onto itself which takes $F$ onto $G$. A corollary of this is a lifting theorem for operators into $L₀/F$, where $F$ is a subspace of $L₀$ isomorphic to the space of all sequences.

Let $L₀$ denote the space of all measurable functions on $[0, 1]$ with the topology of convergence in measure. In [1] it was proved that if $F$ and $G$ are finite-dimensional subspaces of $L₀$ of the same finite dimension, then there is an isomorphism (linear homeomorphism) of $L₀$ onto itself which takes $F$ onto $G$. In this note we prove this result when $F$ and $G$ are isomorphic to $ω$, the space of all sequences.

**THEOREM.** Let $F$ and $G$ be subspaces of $L₀$ which are isomorphic to $ω$. Then there is an isomorphism of $L₀$ onto itself taking $F$ onto $G$.

We give a corollary and then prove the theorem. Recall that an $F$-space $X$ has $L₀$-structure if, for each $ε > 0$, $X$ can be written as a topological direct sum $X = ⊕_{i=1}^{n} X_i$ where each $X_i$ is a subspace of $X$ and the diameter of $X_i$ is less than $ε$, for each $i$ (see [1]).

**COROLLARY.** Let $F$ be a subspace of $L₀$ which is isomorphic to $ω$. Let $X$ be an $F$-space with $L₀$-structure, and let $T$ be a linear operator from $X$ into $L₀/F$. Then there is a unique linear operator $\hat{T}$ from $X$ to $L₀$ such that $T = \pi \hat{T}$, where $\pi$ is the canonical quotient map from $L₀$ onto $L₀/F$. ($\hat{T}$ is said to be a lifting of $T$.)

**PROOF OF THE COROLLARY FROM THE THEOREM.** We first give some notation and describe a special setting of the corollary which will be useful in the proof of the theorem.

If $f$ is in $L₀$, we denote by $[f]$ the one-dimensional space spanned by $f$. The support of $f$ will be denoted by $\text{supp}\ f$. If $A$ is a measurable subset of $[0, 1]$, we denote by $L₀(A)$ the subset of $L₀$ consisting of all $f$ such that $\text{supp}\ f \subset A$. As usual, functions equal almost everywhere are identified, and relations between sets are stated modulo sets of measure zero.

Suppose that $(f_i)$ is a sequence of nonzero elements of $L₀$ such that the sets $S_i = \text{supp}\ f_i$ are pairwise disjoint. Let $F = \text{span}(f_i)$. It is clear that on $F$, the...
$L_0$-topology is equivalent to the topology of convergence in measure on each $S_i$; hence $F$ is isomorphic to $\omega$. We call a copy of $\omega$ obtained in this way a disjoint copy of $\omega$.

Keeping the previous notation, we suppose that $F = \overline{\operatorname{span}(f_i)}$ is a disjoint copy of $\omega$. Then if $B = \bigcup_i \operatorname{supp} f_i$ and if $C = [0, 1] \sim B$, we have that $L_0/F$ is the topological product $L_0(C) \oplus \prod_{i=1}^{\infty} L_0(\operatorname{supp} f_i)/[f_i]$, canonically. Suppose $T: X \to L_0/F$ is a linear operator. Let $q_i$ be the quotient map of $L_0/F$ onto $L_0(\operatorname{supp} f_i)/[f_i]$, for each $i$. By [1, Theorem 3.6], for each $i$ there is a map $\tilde{T}_i: X \to L_0(\operatorname{supp} f_i)$ which is a lifting of the map $q_i T$. Then the map $\chi_C T \oplus \prod_i \tilde{T}_i$ is a lifting of $T$, as required. The uniqueness of $\tilde{T}$ follows immediately: if $T_1$ is another lifting of $T$, then $\tilde{T} - T_1$ maps into the locally convex space $F$, and so must be identically zero.

Now suppose $G$ is any isomorph of $\omega$ in $L_0$. We have shown that the conclusion of the corollary holds for $L_0/F$, above. By the theorem, there is an isomorphism of $L_0$ onto itself which takes $F$ onto $G$; hence the corollary holds in general.

Proof of the theorem.

Lemma 1. Let $F = \overline{\operatorname{span}(f_i)}$ and $G = \overline{\operatorname{span}(g_i)}$ be two disjoint copies of $\omega$ in $L_0$. Then there is an isomorphism of $L_0$ onto itself which takes $F$ onto $G$.

Proof. Let $A = [0, 1] \sim \bigcup_{i=1}^{\infty} \operatorname{supp} f_i$ and let $B = [0, 1] \sim \bigcup_{i=1}^{\infty} \operatorname{supp} g_i$. Let $C = A \cup \operatorname{supp} f_1$ and let $D = B \cup \operatorname{supp} g_1$. By [1, Proposition 2.2] there is an isomorphism $T_1$ of $L_0(C)$ onto $L_0(D)$ taking $f_1$ onto $g_1$. For the same reason, if $i > 2$ there is an isomorphism $T_i$ of $L_0(\operatorname{supp} f_i)$ onto $L_0(\operatorname{supp} g_i)$ taking $f_i$ onto $g_i$. Putting these isomorphisms together in the obvious way yields the result. Q.E.D.

We next consider an intermediate case (the linearly independent case):

Let $(e_j)$ be a sequence in $L_0$ equivalent to the natural basis of $\omega$. In addition, assume that there is a partition $(B_i)$ of $\bigcup_j \operatorname{supp} e_j$ such that for each $i$,

$$N_i = \{j \in N: \operatorname{supp} e_j \cap B_i \neq \emptyset\}$$

is a finite set, and the set of restrictions $\{e_j|_B: j \in N_i\}$ is a linearly independent set. By [1, Proposition 2.2], for each $i$, there is an isomorphism $T_i$ of $L_0(B_i)$ onto itself taking the functions $\{e_j|_B: j \in N_i\}$ onto disjointly supported functions.

Then the maps $T_i$ obviously induce an isomorphism of $L_0(\bigcup B_i)$ onto itself taking the $e_j$'s onto disjointly supported functions. This completes the proof in this case, since it has been reduced to the disjoint case.

Turning now to the general case, let $(e_j)$ be a sequence equivalent to the natural basis of $\omega$, with no additional assumptions. We will prove: there is a basis $(\tilde{e}_j)$ of $\overline{\operatorname{span}(e_j)}$ and a sequence $(B_j)$ of pairwise disjoint measurable sets satisfying

(i) each $\tilde{e}_j$ is a finite linear combination of the $e_j$'s;
(ii) for each $i$, the set $N_i = \{j \in N: \operatorname{supp} \tilde{e}_j \cap B_j \neq \emptyset\}$ is a finite set;
(iii) for each $i$, the set of restrictions $\{\tilde{e}_j|_B: j \in N_i\}$ is linearly independent;
(iv) $\bigcup_j B_j = \bigcup_j \operatorname{supp} \tilde{e}_j$.

Once this has been proved, the argument for the linearly independent case can be applied to the sequences $(\tilde{e}_j)$ and $(B_j)$ and the proof of the theorem will be complete.

We next single out an important property of arbitrary isomorphs of $\omega$ in $L_0$.
Lemma 2. Let $E$ be a subspace of $L_0$ isomorphic to $u$, and let $(e_i)$ be a sequence in $E$ corresponding to the usual basis of $\omega$. Then
\[
\lim_{n \to \infty} \mu\left( \bigcup_{k=n}^{\infty} \text{supp } e_k \right) = 0,
\]
where $\mu$ is Lebesgue measure.

Proof. If the statement is false, there are $\varepsilon > 0$ and a subsequence $(n_k)$ of the positive integers such that
\[
\mu\left( \bigcup_{i=n_k}^{n_k+1-1} \text{supp } e_i \right) > \varepsilon,
\]
for each $k$. By [2, Lemma 1] for each $k$ we can find scalars $a_{n_k}, a_{n_k+1}, \ldots, a_{n_k+1-1}$ such that
\[
\text{supp } \left( \sum_{i=n_k}^{n_k+1-1} a_i e_i \right) = \bigcup_{i=n_k}^{n_k+1-1} (\text{supp } e_i).
\]
Set $g_k = \sum_{i=n_k}^{n_k+1-1} a_i e_i$. Then $\mu(\text{supp } g_k) > \varepsilon$, so we can choose scalars $r_k$ so large that $\int(|r_k g_k|/(1 + |r_k g_k|)) \, d\mu > \varepsilon$. But then we have a contradiction, since $r_k g_k \to 0$. Q.E.D.

Now let the $e_i$'s be as in Lemma 2, and let $\mathcal{F}$ be the family of all finite subsets of the positive integers. For each $F \in \mathcal{F}$, let
\[
C_F = \bigcap_{i \in F} \text{supp } e_i \sim \bigcup_{i \notin F} \text{supp } e_i.
\]
The sets $\{C_F: F \in \mathcal{F}\}$ are pairwise disjoint and (the important point), by Lemma 2, $\mu(\bigcup_{i=1}^{\infty} \text{supp } e_i \sim \bigcup_{F \in \mathcal{F}} C_F) = 0$.

Let $(C_i)$ be an enumeration of the nonempty sets among the sets $C_F, F \in \mathcal{F}$.

We shall describe a repetitive procedure for generating the sequence $\tilde{e}_i$. The procedure alternates between two similar steps in such a way that we are sure to consider every $e_i$ and every $C_i$. During each step, we delete elements from the sequence $(e_i)$ and the enumeration $(C_i)$.

Step $2s - 1$ $(s = 1, 2, \ldots)$. Find the smallest subscript $m$ such that $e_m$ has not been deleted from the sequence $(e_i)$. (In the first application of this step, $m = 1$.) Choose $n$ such that $C_n \cap \text{supp } e_m \neq \emptyset$.

Consider the (finite) set
\[
P_{2s-1} = \{ e_{|C_n}: i \in F_{2s-1} \}
\]
consisting of all restrictions $e_i|_{C_n}$ for which $C_n \cap \text{supp } e_i \neq \emptyset$. (Only $e_i$'s which have not been deleted on a previous step are to be included.) From this set extract a subset $\{ e_{|C_n}: i \in K_{2s-1} \}$ which is a basis for the span of $P_{2s-1}$.

We require that $e_m|_{C_n}$ be one of the elements of this basis.

Let $\tilde{e}_k = e_k$ for $k$ in $K_{2s-1}$, and delete those $e_k$'s from the original sequence $(e_i)$. For each $j$ in $F_{2s-1} \sim K_{2s-1}$, let $g_j$ be the linear combination of the functions $\{ \tilde{e}_k: k \in K_{2s-1} \}$ such that $e_j + g_j \equiv 0$ on $C_n$. For $j$ in $F_{2s-1} \sim K_{2s-1}$, replace each $e_j$ in the original sequence by $e_j + g_j$ (and relabel it $e_j$).
Delete $C_n$ from the original enumeration $(C_i)$, and define $A_{2s-1} = C_n$.

**Step 2s** ($s = 1, 2, \ldots$). Find the smallest subscript $n$ such that $C_n$ has not been deleted from the enumeration $(C_i)$. Delete $C_n$ from the enumeration, and define $A_{2s} = C_n$. Choose an $m$ with $C_n \cap \text{supp } e_m \neq \emptyset$. (If there is no such $m$, set $F_{2s} = K_{2s} = \emptyset$ and terminate this step.) Just as in the odd-numbered step above, consider the set $P_{2s}$, extract a basis, define the corresponding $\tilde{e}_i$'s, delete those elements from $(e_i)$, and replace other elements in $(e_i)$. This ends step 2s.

To generate the complete sequence $(\tilde{e}_i)$, we do step 1, step 2, step 3, \ldots. Notice that after step 2s, 

$$\text{span}\{\tilde{e}_i : \tilde{e}_i \text{ has been defined}\} \supset \text{span}\{e_1, \ldots, e_s\}.$$ 

Thus

(a) span$(\tilde{e}_i) = $ span$(e_i)$.

Also,

(b) $\bigcup_{j=1}^{\infty} A_j = \bigcup_i C_i = \bigcup_i \text{supp } e_i = \bigcup_i \text{supp } \tilde{e}_i$. It is easy to see that

(c) for each $s$, $\{\tilde{e}_i|_{A_s} : i \in K_s\}$ is a linearly independent set of functions; and

(d) $\tilde{e}_i|_{A_s} \equiv 0$ for $i < s$, $i \in K_s$.

From (c) it follows that for each $s$ there is a sequence $(A_s)^{\infty}_{m=1}$ of pairwise disjoint measurable sets which partition $A_s$ and have the property that, for each $m$, the set of restrictions $\{\tilde{e}_j|_{A_s} : j \in K_s\}$ is linearly independent. (This is proved in Lemma 3.) Now define

$$B_i = \bigcup_{j=-1}^{i} A_{i-j+1},$$

for each $i$. Note that $\{B_i\}$ is a partition of $\bigcup_i \text{supp } e_i$. Also note that, for each $i$, only finitely many of the functions $\tilde{e}_j$ are not identically zero on $B_i$—namely, those defined in the $i$th step and possibly some of those defined in earlier steps. Thus conditions (i), (ii), and (iv) are satisfied for the sequences $(\tilde{e}_i)$, $(B_i)$. It remains to check condition (iii).

Let $k$ be an integer and suppose that

$$\sum_{i \in N_k} c_i \tilde{e}_i \equiv 0 \; \text{ on } B_k;$$

we must show that each $c_i$ is zero. Write $N_k = M_1 \cup M_2 \cup \cdots \cup M_k$, where $i$ is in $M_j$ if $\tilde{e}_i$ was defined in step $j$. Then $\sum_{i \in N_k} c_i \tilde{e}_i|_{A_k} \equiv 0$. But on $A_k^i$ all $\tilde{e}_i$ are zero except those defined in step 1. So

$$\sum_{i \in M_1} c_i \tilde{e}_i|_{A_k^1} \equiv 0;$$

since restrictions to $A_k^1$ are linearly independent, it follows that $c_i = 0$ if $i$ is in $M_1$. Next, working on $A_{k-1}^2$, we obtain that

$$\sum_{i \in M_1 \cup M_2} c_i \tilde{e}_i|_{A_{k-1}^2} \equiv 0,$$

and then $\sum_{i \in M_2} c_i \tilde{e}_i|_{A_{k-1}^2} \equiv 0$, and we conclude similarly that $c_i = 0$ for $i$ in $M_2$. Proceeding in this way, we obtain that all the $c_i$'s are 0. This shows that the
sequences \((\tilde{e}_i), (B_i)\) have the properties claimed for them and completes the proof of the theorem.

**Lemma 3.** Let \(A\) be a measurable set of positive measure in \([0, 1]\) and let \((f_i)_{i=1}^n\) be linearly independent elements of \(L_0(A)\). Then there are disjoint measurable subsets \(A_1\) and \(A_2\) of \(A\), each of positive measure, such that \(\{f_i|_{A_1}\}\) and \(\{f_i|_{A_2}\}\) are linearly independent sets.

**Proof.** For a measurable set \(B\) in \(A\) of positive measure, define \(\mathcal{R}_B = \{r = (r_1, r_2, \ldots, r_n) : \Sigma r_i f_i = 0\ \text{a.e. on } B\}\). We first show that given \(B\), there is a measurable \(C \subset B\) of positive measure such that if \(D \subset C\) and \(D\) has positive measure, then \(\mathcal{R}_D = \mathcal{R}_C\). To see this, given \(B\), choose, if possible, \(B_1 \subset B\), \(B_1\) of positive measure, and a vector \(r_1\) in \(\mathcal{R}_{B_1}\), \(r_1 \neq 0\). Now choose, if possible, a set \(B_2 \subset B_1\), \(B_2\) of positive measure, and a vector \(r_2\) in \(\mathcal{R}_{B_2}\), \(r^2\) independent of \(r^1\). Now choose, if possible, a set \(B_3 \subset B_2\), \(B_3\) of positive measure, and \(r_3\) in \(\mathcal{R}_{B_3}\), \(r^3\) independent of \(r^1\) and \(r^2\). Continue. This process must terminate with some \(B_j\), \(j < n\). Thus if \(B_j = C\) and \(D \subset C\), \(D\) of positive measure, then \(\mathcal{R}_D = \mathcal{R}_C\).

Continuing with the proof of the lemma, we let \(\mathcal{G}\) be the family of all measurable subsets \(C\) of \(A\) with positive measure and having the property that if \(D\) is a measurable subset of \(C\) of positive measure, then \(\mathcal{R}_D = \mathcal{R}_C\). The above construction shows that every measurable set in \(A\) of positive measure contains a set in \(\mathcal{G}\). Now let \(\mathcal{C}\) be a maximal family of pairwise disjoint elements of \(\mathcal{G}\). Then \(\mathcal{C}\) is countable. Let \((C_i)\) be an indexing of the elements of \(\mathcal{C}\); by maximality, \(A \sim \bigcup_i C_i\) has measure zero.

For each \(i\) let \(E_i\) be any measurable subset of \(C_i\) with \(0 < \mu(E_i) < \mu(C_i)\) and let \(F_i = C_i \sim E_i\). Let \(A_1 = \bigcup_i E_i\) and let \(A_2 = \bigcup_i F_i\). Suppose \(r\) is a nonzero vector in \(\mathcal{R}_{A_1}\). Then \(r \in \mathcal{R}_{E_i}\) for each \(i\), so \(r \in \mathcal{R}_{C_i}\) and then \(r \in \mathcal{R}_{A_2}\), contradicting the linear independence of \((f_i)\) on \(A\). Similarly, \(\mathcal{R}_{A_2}\) contains the 0 vector alone. The proof of the lemma is complete.

**References**

1. N. J. Kalton and N. T. Peck, *Quotients of \(L_0(0, 1)\) for \(0 < p < 1\)*, Studia Math. 64 (1979), 65–75.

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