CONTRACTIVE COMMUTANTS AND INVARIANT SUBSPACES

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Abstract. Let $T$ be a bounded operator on a Banach space $\mathcal{X}$ and let $K$ be a nonzero compact operator. In [1] and [4] it is shown that if $\lambda$ is a complex number and if $TK = \lambda KT$, then $T$ has a hyperinvariant subspace. In [1], S. Brown goes on to show that if $\mathcal{X}$ is reflexive and if $TK = \lambda KT$ and $TB = \mu BT$ for some $\lambda, \mu$ with $|\lambda| \neq 1$ and $(1 - |\mu|)/(1 - |\lambda|) > 0$, then $B$ has an invariant subspace. Below we extend both these results by showing that the entire class of operators satisfying the above conditions on $B$ has an invariant subspace.

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1. The contractive commutant. Let $\mathcal{X}$ be an infinite-dimensional Banach space and let $\mathcal{L}(\mathcal{X})$ be the algebra of bounded linear operators on $\mathcal{X}$. The commutant $\{T\}'$ of an operator $T$ is the algebra of operators $B$ that commute with $T$. A basic result is the elegant theorem of Lomonosov [5]; the statement below is the distillation by Pearcy and Shields [6].

Theorem 1.1 (Lomonosov). If $\alpha$ is a subalgebra of $\mathcal{L}(\mathcal{X})$ with no nontrivial invariant subspaces, and if $K$ is any nonzero compact operator, then there is an operator $A$ in $\alpha$ such that $1$ is an eigenvalue of $AK$.

Corollary 1.2 (Lomonosov). If $\{T\}'$ contains a nonzero compact operator and $T$ is not a scalar multiple of the identity, then $\{T\}'$ has an invariant subspace.

Let $\mathcal{C}_c(T) = \{B \in \mathcal{L}(\mathcal{X}): TB = \lambda BT\}$. Notice that $\mathcal{C}_c(T)$ is not an algebra, since it fails to be closed under sums. Let $\{T\}'_c$ be the (nonclosed) algebra generated by $\mathcal{C}_c(T)$. We refer to $\{T\}'_c$ as the contractive commutant of $T$. Similarly, let $\mathcal{C}_{\infty}(T) = \{B \in \mathcal{L}(\mathcal{X}): TB = \lambda BT\}$ and let $\{T\}'_{\infty}$ be the algebra generated by $\mathcal{C}_{\infty}(T)$; we call $\{T\}'_{\infty}$ the strictly contractive commutant of $T$. A number of simple facts are listed below.

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Theorem 1.3. (i) If $A, B$ are in $C_c(T)$ (resp. $C_{sc}(T)$), and if $\mu \in C$ then $\mu A$ and $AB$ are in $C_c(T)$ (resp. $C_{sc}(T)$).

(ii) If $A \in C_c(T)$ and $B \in C_{sc}(T)$ then $BA$ and $AB$ lie in $C_{sc}(T)$.

(iii) If $A \in C_c(T)$ (resp. $C_{sc}(T)$) then $T^* \in C_c(A^*)$ (resp. $C_{sc}(A^*)$).

(iv) $C_c(T)$ is closed in the weak operator topology.

Proof. (i), (ii), and (iii) are straightforward computations. To prove (iv) we suppose that $\{B_\alpha\}$ is a net of operators in $C_c(T)$ and $B_\alpha \to B$ weakly. If $\lambda_\alpha$ is chosen so that $TB_\alpha = \lambda_\alpha B_\alpha T$, then $|\lambda_\alpha| < 1$ for all $\alpha$ and thus there is a convergent subnet of $\{\lambda_\alpha\}$; without loss of generality we assume that the entire net $\{\lambda_\alpha\}$ converges, say to $\lambda$. Then $TB_\alpha$ converges weakly to $TB$, $\lambda_\alpha B_\alpha T$ converges weakly to $\lambda BT$, and the result follows.

Lemma 1.4. $\{T\}'_c$ (resp. $\{T\}'_{sc}$) consists precisely of finite sums, $\sum_{i=1}^n B_i$, where each $B_i$ lies in $C_c(T)$ (resp. $C_{sc}(T)$).

Proof. $\{T\}'_c$ is the algebra generated by $C_c(T)$, so clearly every finite sum of operators in $C_c(T)$ belongs to $\{T\}'_c$. It is easy to check, using 1.3(i), that the collection of finite sums is an algebra, and thus that it is the same as $\{T\}'_c$. The statement for $\{T\}'_{sc}$ follows similarly.

Corollary 1.5. If $A \in \{T\}'_c$ and $B \in \{T\}'_{sc}$, then $AB$ and $BA$ lie in $\{T\}'_{sc}$.

Proof. Use Lemma 1.4 and Theorem 1.3(iii).

The proof of the next result is a slight sharpening of the proof of Theorem 2 of [1].

Theorem 1.6. (i) If $TB = \lambda BT$ for some complex number $\lambda$ (not necessarily in the unit disk) then either $|\lambda| = 1$ or $TB$ and $BT$ are quasinilpotent.

(ii) If $TK = \lambda KT$ where $K$ is compact then either $\lambda$ is a root of unity or $TK$ and $KT$ are quasinilpotent.

Proof. (i) It is well known that the nonzero elements of $\sigma(TB)$ and $\sigma(BT)$ are the same [3, p. 63]. Thus it follows that $r(TB) = r(BT)$, where $r(X)$ denotes the spectral radius of $X$. Since $TB = \lambda BT$ we also have $r(TB) = |\lambda|r(BT)$ and thus $r(BT) = |\lambda|r(BT)$. Hence either $|\lambda| = 1$ or else $r(BT)$ (and therefore $r(TB)$) is 0.

(ii) Suppose $TK = \lambda KT$ and $TK$ and $KT$ are not quasinilpotent. By part (i), $|\lambda| = 1$. Let $0 \neq z \in \sigma(KT)$. Then $\lambda z \in \sigma(TK) = \sigma(KT)$. By induction, $\lambda^n z \in \sigma(KT)$ for all nonnegative integers $n$. However, $KT$ is compact and its spectrum cannot contain an infinite set of numbers whose absolute values are bounded away from 0. Thus $\{\lambda^n z\}_{n=0}^\infty$ is a finite set and $\lambda$ must be a root of unity.

2. Invariant subspaces. The contractive commutant contains the commutant; thus it is less likely that the former should have nontrivial invariant subspaces. The following example shows that $\{T\}'_c$ may indeed be transitive.

Example 2.1. Let $\mathcal{H}$ be a two-dimensional Hilbert space and let $T, K_1$ and $K_2$ be defined by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad K_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
Then $K_1$ and $K_2$ both lie in $C_c(T)$, but it is easy to see that no subspace is invariant for $T, K_1,$ and $K_2$. Notice that in this case, $(T)_c = \mathcal{L}(\mathcal{H})$.

For an infinite-dimensional example, let $\mathcal{H}$ be any Hilbert space and let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$. Let $T = \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H})$. $C_c(T)$ contains all operators of the form

\[
\begin{pmatrix}
A & 0 \\
0 & C
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 \\
0 & D
\end{pmatrix}
\]

Thus $(T)_c = \mathcal{L}(\mathcal{H})$ in this case as well. Observe that if $(T)_c$ were transitive and not dense in $\mathcal{L}(\mathcal{H})$ we would have a solution to the transitive algebra problem.

Our central result shows that under certain conditions $(T)_c$ and $(T)_sc$ do have invariant subspaces.

**Theorem 2.2.** Let $T$ be a nonzero operator in $\mathcal{L}(\mathcal{H})$. If $(T)_sc$ (resp. $(T)_c$) contains a nonzero compact operator, then $(T)_c$ (resp. $(T)_sc$) has a nontrivial invariant subspace.

**Proof.** Let $(T)_sc$ contain a nonzero compact operator $K$. Note that ker $F$ is an invariant subspace for $C_c(T)$ and hence for $(T)_c$; we therefore assume that ker $T = \{0\}$. Suppose that $(T)_c$ is a transitive algebra. Theorem 1.1 guarantees the existence of an operator $B$ in $(T)_c$ and a nonzero vector $x$ such that $BKx = x$. By Corollary 1.5, $BK \in (T)_sc$ and thus there exist $B_1, \ldots, B_n \in C_c(T)$ such that $\sum_{i=1}^{n} B_i = BK$. Let $TB_i = \lambda_i B_i T$, where $|\lambda_i| < 1$ for each $i$. Then $TBK = \sum TB_i = (\sum \lambda_i B_i) T$ and inductively $T^m BK = (\sum_{i=1}^{n} \lambda_i^m B_i) T^m$ for each positive integer $m$. Hence $T^m x = T^m BKx = (\sum_{i=1}^{n} \lambda_i^m B_i) T^m x$. We have assumed that $T$ has trivial kernel and thus $T^m x \neq 0$ for every $m$, and it follows that 1 lies in the point spectrum of $\sum_{i=1}^{n} \lambda_i^m B_i$ for every $m$. However, this would imply that $1 < ||\sum_{i=1}^{n} \lambda_i^m B_i|| < ||\sum_{i=1}^{n} ||\lambda_i||^m ||B_i||$ for all $m$, which is obviously impossible since $|\lambda_i| < 1$ for all $i$. Hence the assumption that $(T)_c$ is transitive must be false.

The proof of the other part of the theorem is virtually identical and is omitted.

The corollary is a generalization of Theorem 3 of [1].

**Corollary 2.3.** Suppose that $\mathcal{H}$ is reflexive and that $TK = \lambda KT$ for $K$ a nonzero compact operator, $T$ nonzero, and $|\lambda| \neq 1$. Let $a$ be the algebra generated by all operators $B$ such that $TB = \mu BT$ for some complex number $\mu$ for which $(1 - |\mu|)/(1 - |\lambda|) > 0$. Then $a$ has an invariant subspace.

**Proof.** The theorem covers the case $|\lambda| < 1$. If $|\lambda| > 1$ then $T^* K^* = \lambda^{-1} K^* T^*$ and $K^* \in C_c(T^*)$. The theorem then shows that $(T^*)_c$ has an invariant subspace. Note that $a = \{ B : B^* \in \{ T^* \}_c \}$. Hence $a^*$ has an invariant subspace, and because of the reflexivity of $\mathcal{H}$ so does $a$.

**Question.** Is it possible to show the existence of an invariant subspace for $(T)_c$ under the weaker assumption that the closure (in some appropriate topology) of $(T)_sc$ contains a nonzero compact operator? A reasonable first step might be to show that the weaker condition yields a hyperinvariant subspace for $T$.

We remark that C. K. Fong [2] has recently obtained some related results concerning common invariant subspaces of $T$ and $K$, under more general conditions than those discussed here.
References


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