A MONOTONIC PROPERTY FOR THE ZEROS OF ULTRASPHERICAL POLYNOMIALS

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Abstract. It is shown that $\lambda x_{n,k}^{(\lambda)}$ increases as $\lambda$ increases for $0 < \lambda < 1$, $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$, where $x_{n,k}^{(\lambda)}$ is the $k$th positive zero of ultraspherical polynomial $P_n^{(\lambda)}(x)$.

The aim of this work is to prove the following

Theorem. Let $x_{n,k}^{(\lambda)}$, $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$, be the zeros of the ultraspherical polynomial $P_n^{(\lambda)}(x)$ in decreasing order on $(0, 1)$, where $0 < \lambda < 1$.

Then for every $\epsilon > 0$,

$$x_{n,k}^{(\lambda)} < (\lambda + \epsilon)x_{n,k}^{(\lambda+\epsilon)}, \quad k = 1, 2, \ldots, \left[\frac{n}{2}\right].$$

Remark 1. For our purposes the following form of Sturm comparison theorem will prove useful. This formulation differs from the usual formulation [2, p. 19] in that $f(x) < F(x)$ is hypothesized for the interval $a < x < X_m$, rather than for the larger interval $a < x < x_m$. (See the work [1] for the proof of this formulation of Sturm theorem.)

Lemma. Let the functions $y(x)$ and $Y(x)$ be nontrivial solutions of the differential equations

$$y''(x) + f(x)y(x) = 0; \quad Y''(x) + F(x)Y(x) = 0$$

and let them have consecutive zeros at $x_1, x_2, \ldots, x_m$ and $X_1, X_2, \ldots, X_m$ respectively on an interval $(a, b)$. Suppose that $f(x)$ and $F(x)$ are continuous, that $f(x) < F(x)$, $a < x < X_m$, and that

$$(1) \quad \lim_{x \to a^+} \left[ y'(x)Y(x) - y(x)Y'(x) \right] = 0.$$ 

Then

$$X_k < x_k, \quad k = 1, 2, \ldots, m.$$ 

Proof of the theorem. The function

$$u(x) = (1 - x^2)^{\lambda/2 + 1/4} P_n^{(\lambda)}(x)$$

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which has the same zeros on \((-1, 1)\) as \(P_n^{(\lambda)}(x)\) satisfies [2, p. 82] the differential equation
\[
y''(x) + p_\lambda(x)y(x) = 0
\]
where
\[
p_\lambda(x) = \frac{(n + \lambda)^2}{1 - x^2} + \frac{2 + 4\lambda - 4\lambda^2 + x^2}{4(1 - x^2)^2}.
\]
The functions \(u(x/\lambda)\) and \(u(x/(\lambda + \varepsilon))\) that have on the interval \((0, \lambda)\) and \((0, \lambda + \varepsilon)\) consecutive zeros at \(\lambda x_{n,1}^{(\lambda)}, \ldots, \lambda x_{n,2}^{(\lambda)}\) and \((\lambda + \varepsilon)x_{n,1}^{(\lambda + \varepsilon)}, \ldots, (\lambda + \varepsilon)x_{n,2}^{(\lambda + \varepsilon)}\) respectively, satisfy the differential equations
\[
z''(x) + \psi_\lambda(x)z(x) = 0, \quad W''(x) + \psi_{\lambda + \varepsilon}(x)w(x) = 0
\]
where \(\psi_\lambda(x) = \nu^{-2}p_\lambda(x)\).

It is easy to show that \(\psi_\lambda(x)\) decreases as \(\lambda\) increases for \(0 < \lambda < 1\) and \(0 < x < \lambda\), that is \(\psi_\lambda(x) > \psi_{\lambda + \varepsilon}(x), \varepsilon > 0\). The limit condition (1) is easily shown to hold in the present case. Thus the above Lemma is applicable and its conclusion gives the desired result.

**Remark 2.** Our result contrasts with \(x_{n,k}^{(\lambda)} > x_{n,k}^{(\lambda + \varepsilon)}\) which follows from formula
\[
\partial x_{n,k}^{(\lambda)}/\partial \lambda < 0, \quad k = 1, 2, \ldots, \left[\frac{n}{2}\right].
\]
Putting this result together with the new result gives
\[
1 < \frac{x_{n,k}^{(\lambda)}}{x_{n,k}^{(\lambda + \varepsilon)}} < 1 + \frac{\varepsilon}{\lambda}, \quad k = 1, 2, \ldots, \left[\frac{n}{2}\right].
\]

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**References**

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