A NOTE ON A LEMMA OF SHELAH
CONCERNING STATIONARY SETS

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Abstract. Let $\kappa$ be an infinite cardinal, let $I$ be a nonprincipal ideal on $\kappa$ and let $I^+ = \{ X \subseteq \kappa: X \not\in I \}$. $S(I)$ is the following property of ideals: for every $A \in I^+$ and every pair of functions $f, g$ from $A$ into $\kappa$ such that, for every $\alpha \in A$, $f(\alpha) \neq g(\alpha)$, there exists a set $B \subseteq A$ with $B \in I^+$ such that $f''B \cap g''B = \emptyset$. We prove that $S(I)$ holds for every weakly selective ideal $I$ on any infinite cardinal $\kappa$ (including $\kappa = \omega$), and that $S(I)$ holds for every $\kappa$-complete ideal on $\kappa$ if $\kappa$ is not strongly inaccessible.

Let $\kappa$ be an infinite cardinal. A (proper) ideal on $\kappa$ is a collection $I$ of subsets of $\kappa$ such that $\kappa \in I$ and whenever $X, Y \in I$ and $Z \subseteq X \cup Y$, then $Z \in I$. If $I$ is an ideal on $\kappa$ then $I^+$ denotes the sets of “positive $I$-measure”; i.e. $I^+ = \{ X \subseteq \kappa: X \not\in I \}$. $S(I)$ is the following property of ideals: for every $A \in I^+$ and every pair of functions $f, g$ from $A$ into $\kappa$ such that, for every $\alpha \in A$, $f(\alpha) \neq g(\alpha)$, there exists a set $B \subseteq A$ with $B \in I^+$ such that $f''B \cap g''B = \emptyset$. Shelah's lemma [EM] is the assertion $S(\operatorname{NS}_{\kappa})$, where $\operatorname{NS}_{\kappa}$ is the ideal of nonstationary subsets of the regular uncountable cardinal $\kappa$. The following result will provide a short proof of a generalization of Shelah's lemma.

Theorem 1. Let $S'(I)$ denote the weaker version of $S(I)$ obtained by considering only functions $f$ and $g$ that are one-to-one. Then $S'(I)$ holds for every ideal $I$ on every infinite cardinal $\kappa$ (including $\kappa = \omega$).

Proof. Let $G$ be the graph on $A$ obtained by making $\alpha$ adjacent to $\beta$ (where $\alpha < \beta$) iff $g(\alpha) = f(\beta)$. Then each point $B \in A$ is adjacent to at most one $\alpha < \beta$ (since otherwise we would have $f(\beta) = g(\alpha_1)$ and $f(\beta) = g(\alpha_2)$ contradicting the one-to-oneness of $g$). Thus each $\beta \in A$ gives rise to a unique decreasing path of finite length. Without loss of generality, assume that the set $B'$ of points $\beta \in A$ having such a path of even length is of positive $I$-measure. Since $B'$ is clearly an independent set in the graph $G$ it follows that if we have $\alpha, \beta \in B'$ with $\alpha < \beta$ then $g(\alpha) \neq f(\beta)$. Now we simply repeat the procedure (starting with $B'$) with the roles of $f$ and $g$ reversed. The set $B \subseteq B'$ of positive $I$-measure so obtained clearly has the property that $f''B \cap g''B = \emptyset$ as desired.

Remark. It is worth noting that we really do not need both $f$ and $g$ to be one-to-one—just the “larger.” That is, if we let $A_{\alpha} = \{ \alpha \in A: f(\alpha) < g(\alpha) \}$ and

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764

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Af = {a ∈ A: g(a) < f(a)} then either Af ∈ I⁺ or Ag ∈ I⁺. If, for example, Af ∈ I⁺ then we can redo the second step in the above proof so as to appeal to this fact instead of the one-to-oneness of f as follows. Let G' be the graph on B' in which α is adjacent to β (where α < β) iff f(α) = g(β). Then each α is adjacent to at most one β > α (since g is one-to-one). Notice also that if α < β and α is adjacent to β then g(α) > g(β); that is, g(β) = f(α) < g(α). Hence each α gives rise to a unique increasing path of finite length and so we can proceed exactly as in the first part of the proof of Theorem 1.

Recall that an ideal I on κ is said to be normal if every regressive function f (i.e. f(α) < α for α ≠ 0) defined on a set of positive I-measure is constant on a set of positive I-measure. (Fodor's theorem [F] asserts that NSκ is normal if κ is a regular uncountable cardinal.) I is said to be weakly selective if every function defined on a set of positive I-measure or one-to-one on a set of positive I-measure. Weglorz first observed that every normal ideal I is weakly selective. (In fact, if I is normal, A ∈ I⁺, f: A → κ and f⁻¹(α)) ∈ I for every α < κ, then the set B = A − {inf(f⁻¹(α)): α < κ} is in I as can be seen by considering the regressive function h: B → κ given by h(α) = inf(f⁻¹(α))). Even on uncountable cardinals there are lots of weakly selective ideals that are not normal (e.g. {A, B, kⁿ+: |A| < kⁿ}; for more see [BTW]). With this much said, an easy consequence of Theorem 1 is the following.

**COROLLARY.** S(I) holds for every weakly selective ideal I on any infinite cardinal κ (including κ = ω).

Theorem 1 and its corollary suggest the possibility that perhaps S(I) holds for every ideal I. This, however, is easily seen not to be the case. For example, if D is an ultrafilter on κ and I is the ideal on κ × κ dual to D × D, then the projection functions show that S(I) fails. These considerations also show that if κ is a measurable cardinal then there is a κ-complete ideal I (that is, one closed under unions of size less than κ) for which S(I) fails. On the other hand, one can use Theorem 1 (and the remark following it) to show that if κ is an infinite successor cardinal then S(I) holds for every κ-complete ideal on κ. Hence, if we momentarily agree to call κ good iff S(I) holds for every κ-complete ideal on κ, then we have that successor cardinals are good and measurable cardinals are not. Our next result will fill the obvious gap (i.e., it will follow that κ is good iff κ is not strongly inaccessible).

**THEOREM 2.** For infinite cardinals κ and μ, the following are equivalent:

(i) \[ κ → \left( \begin{array} \{ \mu \} \end{array} \right)^2 ; \]

i.e., for every f: [κ]² → λ where λ < μ, there exists α, β, γ such that α < β < γ < κ and f((α, β)) = f((β, γ)).

(ii) S(I) fails for some proper nonprincipal μ-complete ideal I on κ.
Proof. (i) \(\rightarrow\) (ii). Assume that
\[
\kappa \rightarrow \left(1\right)^2_{\prec \mu}
\]
and let \(A = \{(\alpha, \beta): \alpha < \beta < \kappa\}\). We will construct a \(\kappa\)-complete proper nonprincipal ideal \(I\) on the set \(A\) so that the projection functions \(\pi_1\) and \(\pi_2\) show that \(S(I)\) fails; this clearly suffices.

Let \(\mathcal{S} = \{X \subseteq A: \pi_1(X) \cap \pi_2(X) = 0\}\) and let \(I\) be the \(\mu\)-complete ideal on \(A\) generated by \(\mathcal{S}\) (i.e., \(Y \in I\) iff \(Y \subseteq \bigcup H\) for some \(H \subseteq \mathcal{S}\) with \(|H| < \mu\)). Then \(I\) is clearly closed downward (i.e., \(Y \subseteq X \in I \Rightarrow Y \in I\)) and under unions of size less than \(\mu\). Moreover, every singleton subset \(\{(\alpha, \beta)\}\) of \(A\) is in \(I\) (since \(\alpha \neq \beta\)). Hence, it remains only to show that \(I\) is proper.

Suppose not. Then \(A = \bigcup \{A_\xi: \xi < \lambda\}\) for some \(\lambda < \mu\) where we have \(A_\xi \in \mathcal{S}\) for each \(\xi < \lambda\). We can assume that the \(A_\xi\)'s are pairwise disjoint. Define \(f: [\kappa]^2 \rightarrow \lambda\) by \(f(\{(\alpha, \beta)\}) = \xi\) if \(\alpha < \beta\) and \((\alpha, \beta) \in A_\xi\). Since \(\lambda < \mu\) and
\[
\kappa \rightarrow \left(1\right)^2_{\prec \mu}
\]
we get some \(\xi < \lambda\) and \(\alpha < \beta < \gamma\) so that \(f(\{(\alpha, \beta)\}) = \xi = f((\beta, \gamma))\). But then \((\alpha, \beta) \in A_\xi\) and \((\beta, \gamma) \in A_\xi\) so \(\beta \in \pi^\kappa_1 A_\xi \cap \pi^\kappa_2 A_\xi\). This contradicts the fact that \(A_\xi \in \mathcal{S}\) and thus shows that \(I\) is proper.

(ii) \(\rightarrow\) (i). Suppose that \(h: [\kappa]^2 \rightarrow \lambda\) for some \(\lambda < \mu\) and \(h\) shows that
\[
\kappa \rightarrow \left(1\right)^2_{\prec \mu}
\]
Let \(I\) be a proper \(\mu\)-complete ideal on \(\kappa\) and suppose that \(f, g: A \rightarrow \kappa\) where \(A \in I^+\) and \(f(\alpha) \neq g(\alpha)\) for every \(\alpha \in A\). For each \(\xi < \lambda\) let \(A_\xi\) be given by
\[
A_\xi = \{\alpha \in A: h(\{(f(\alpha), g(\alpha))\}) = \xi\}.
\]
Since \(I\) is \(\mu\)-complete, \(\lambda < \mu\) and \(A \in I^+\) we get that \(A_\xi \in I^+\) for some \(\xi < \lambda\). Without loss of generality assume that \(B \in I^+\) where \(B = \{\alpha \in A_\xi: f(\alpha) < g(\alpha)\}\).

Now, to complete the proof it suffices to show that \(f''B \cap g''B = \emptyset\).

Suppose not, and choose \(\alpha, \gamma \in B\) such that \(f(\alpha) = g(\gamma) = \beta\). Then \(f(\gamma) < g(\gamma) = \beta = f(\alpha) < g(\alpha)\) and so \(f(\gamma) < \beta < g(\alpha)\). But \(h(\{(f(\gamma), \beta)\}) = h(\{(f(\gamma), g(\gamma))\}) = \xi = h(\{(f(\alpha), g(\alpha))\}) = h(\{(\beta, g(\alpha))\})\) and so the set \(\{f(\gamma), \beta, g(\alpha)\}\) contradicts the fact that \(h\) shows
\[
\kappa \rightarrow \left(1\right)^2_{\prec \mu}.
\]

Corollary. For regular cardinals \(\kappa\) and \(\mu\), the following are equivalent:
(i) \(2^\lambda \geq \kappa\) for some \(\lambda < \mu\).
(ii) \(S(I)\) holds for every \(\mu\)-complete proper ideal \(I\) on \(\kappa\).
Proof. (i) → (ii). Assume that $\lambda < \mu$ and $2^\lambda > \kappa$. By the previous theorem it suffices to show that

$$\kappa \rightarrow \left( \begin{array}{c} \lambda \\ \mu \end{array} \right)^2;$$

our argument here is only a slight (but necessary) modification of the standard example (due to Erdős and Rado [ER]) showing that $2^\lambda \rightarrow (3)_\lambda^2$. So let $h: \kappa \rightarrow \lambda^2$ be one-to-one where $\lambda^2$ denotes the set of all functions mapping $\lambda$ to 2. Define $f$: $[\kappa]^2 \rightarrow \lambda \times 2$ as follows. If $\alpha < \beta$ then set $f(\{\alpha, \beta\}) = (\gamma, i)$ where

$$\gamma = \inf\{\xi < \lambda: h(\alpha)(\xi) \neq h(\beta)(\xi)\}$$

and $h(\alpha)(\gamma) = i$. Now, suppose for contradiction that $\alpha < \beta < \delta$ and $f(\{\alpha, \beta\}) = (\gamma, i) = f(\{\beta, \delta\})$. Without loss of generality, assume that $i = 0$. Then $h(\alpha)(\gamma) = 0$ and $h(\beta)(\gamma) = 1$ (since $f(\{\alpha, \beta\}) = (\gamma, 0)$). But then since $f(\{\beta, \delta\}) = (\gamma, 0)$ we have $h(\beta)(\gamma) = 0$; contradiction.

(ii) → (i). The Erdős-Rado Theorem [ER] asserts that $(2^\lambda)^+ \rightarrow (\lambda^+)^3_\lambda$; it follows trivially from this that if $\kappa > 2^\lambda$ for every $\lambda < \mu$ then

$$\kappa \rightarrow \left( \begin{array}{c} \lambda \\ \mu \end{array} \right)^2.$$ 

The desired result thus follows from the previous theorem. □

Remark. A consequence of the above is that if $\kappa = \sup\{(2^\lambda)^+: \lambda < \kappa\}$ and $\kappa$ is regular, then $S(I)$ fails for some proper nonprincipal $\mu$-complete uniform ideal $I$ on $\kappa$. (To say that $I$ is uniform means that $\{X \subseteq \kappa: |X| < \kappa \} \subseteq I$.)

Corollary. $S(I)$ holds for every $\kappa$-complete proper nonprincipal ideal $I$ on $\kappa$ iff $\kappa$ is not strongly inaccessible.

We conclude with an easy application of the corollary to Theorem 1. An ultrafilter $\mathcal{U}$ on $\kappa$ is said to be Ramsey if every function $f: \kappa \rightarrow \kappa$ is either constant or a set in $\mathcal{U}$ or one-to-one on a set in $\mathcal{U}$. If $\mathcal{U}$ is an ultrafilter on $\kappa$ and $A$ is a set then a subset $X$ of $A^*/\mathcal{U}$ is called standard if there is a $B \subseteq A$ such that $X = B^*/\mathcal{U}$. We claim that if $\mathcal{U}$ is a Ramsey ultrafilter on $\kappa$, then any two elements of $A^*/\mathcal{U}$ can be separated by a standard set. That is, if $[f], [g] \in A^*/\mathcal{U}$ and $[f] \neq [g]$, then the corollary to Theorem 1 yields a set $X \in \mathcal{U}$ so that $f''X \cap g''X = \emptyset$. But now if $B = f''X$, then $[f] \in B^*/\mathcal{U}$ and $[g] \notin B^*/\mathcal{U}$. This application has consequences for certain problems involving cardinalities of ultra-powers; these will appear elsewhere.

References


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