SELECTION AND REPRESENTATION THEOREMS FOR
σ-COMPACT VALUED MULTIFUNCTIONS

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Abstract. In this paper we give two applications of results of Shchegolkov and
Saint-Raymond on Borel sets with σ-compact sections. First we give a sufficient
condition under which a partition of a Polish space into σ-compact sets admits a
Borel cross-section. Then a representation theorem for σ-compact valued multi-
functions, expressing them as unions of continuously indexed Borel graphs, is
proved.

1. Introduction. The past few years have seen much progress in selection theory.
It has been shown that multifunctions whose values are topological nice subsets of
a Polish space (e.g. countable, closed, Gδ, nonmeager) admit a measurable selector.
More generally, they can be represented as the union of graphs of measurable
selectors which are themselves nicely indexed. For details we refer the reader to
Wagner [7, 8].

In this paper we study multifunctions whose values are σ-compact subsets of a
Polish space. Some positive results on such multifunctions were proved in the late
30's and early 40's by Kunugui, Novikov, Arsenin and Shchegolkov. Our main
results can be stated as follows.

Theorem 1.1. Let \( Q \) be a partition of a Polish space \( T \) into σ-compact sets. Then
the following conditions are equivalent:

(i) The σ-field \( A(Q) \) of all Borel sets in \( T \) which are unions of elements of \( Q \)
is countably generated.

(ii) The equivalence relation \( R(Q) \) induced by \( Q \) belongs to \( A(Q) \otimes B_T \), where \( B_T \)
is the Borel σ-field of \( T \).

Moreover if (i) or (ii) holds then \( Q \) admits a Borel cross-section.

Theorem 1.2. Let \( T \) be an analytic space and \( X \) a Polish space. If \( F: T \to X \) is a
σ-compact valued multifunction such that \( \text{graph}(F) \in B_{T \times X} \) then there is a map \( f: T \times (\omega \times C) \to X \) such that

(i) for each \( t \in T \), \( f(t, \cdot) \) is a continuous function from \( \omega \times C \) onto \( F(t) \), and

(ii) for \( m \in \omega \) and \( \alpha \in C \), the map \( t \to f(t, m, \alpha) \) is a Borel measurable selector for
\( F \),

where \( \omega \) is the space of natural numbers and \( C \) denotes the Cantor space \( \{0, 1\}^\omega \).

Received by the editors November 20, 1980.


Key words and phrases. Multifunction, partition, selector, cross-section, representation.

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0002-9939/81/0000-0573/$02.50

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The paper is organised as follows. In §2 we introduce the basic definitions and notation and record some preliminary results. A proof of Theorem 1.1 is given in §3. Theorem 1.2 is established in §4.

The results of this paper are included in my thesis. I express my indebtedness to Professor A. Maitra for his guidance.

2. Preliminaries. A second countable completely metrizable topological space is called a Polish space. We denote the set of natural numbers by \( \omega \), the Cantor set \( (0, 1)^\omega \) by \( C \) and the Baire space \( \omega^\omega \) of infinite sequences of natural numbers will be denoted by \( \Sigma \). A metrizable space \( T \) is called analytic if it is a continuous image of \( \Sigma \). If \( X \) is a Polish space then \( A \subset X \) is called coanalytic if \( X \setminus A \) is analytic. If \( T \) is a metrizable space, \( \mathcal{B}_T \) will stand for the Borel \( \sigma \)-field of \( T \), i.e. \( \mathcal{B}_T \) is the \( \sigma \)-field generated by open sets in \( T \). For results in descriptive set theory we refer the reader to Kuratowski [2].

Let \((F, A)\) and \((S, B)\) be measurable spaces. Then \( A \otimes B \) denotes the product of the \( \sigma \)-fields \( A \) and \( B \). An \( A \)-atom is a nonempty set \( A \in A \) such that \( \emptyset \neq B \subseteq A \) and \( B \in A \). If \( E \subseteq T \times S \) and \( t \in T \) then \( E_t \) denotes the set \( \{s \in S: (t, s) \in E\} \) and is called the section of \( E \) at \( t \). We denote the projection map from \( T \times S \) onto \( T \) by \( \pi_T \). If \( E \subseteq T \times S \) and \( t \in T \) then \( E_t \) stands for the \( \pi_T \)-image of \( E \) at \( t \).

Let \( (T, A) \) be a measurable space and \( A' \) be a Polish space. A multifunction \( F: T \rightarrow X \) is a map whose domain is \( T \) and whose values are nonempty subsets of \( X \). For \( E \subseteq X \), \( F^{-1}(E) \) denotes the set \( \{t \in T: F(t) \cap E \neq \emptyset\} \) by \( F^{-1}(E) \). We say that \( F \) is \( A \)-measurable if \( F^{-1}(U) \in A \) for all open sets \( U \) in \( X \). The set \( \{ (t, x) \in T \times X: x \in F(t) \} \) is called the graph of \( F \) and will be denoted by \( \text{graph}(F) \). A point map \( f: T \rightarrow X \) is called a selector for \( F \) if \( f(t) \in F(t) \) for all \( t \in T \).

A collection \( Q \) of pairwise disjoint, nonempty subsets of \( X \) whose union is \( X \) is called a partition of \( X \). For \( E \subseteq X \), \( E^* \) or simply \( E \) denotes the set \( \bigcup \{A \in Q: A \cap E \neq \emptyset \} \) and is called \( Q \)-saturation of \( E \). We say that \( E \) is \( Q \)-saturated if \( E = E^* \). If \( X \) is a Polish space then \( Q \) is measurable if \( U^* \) is Borel for all open sets \( U \) in \( X \). Also, \( A(Q) \) will denote the \( \sigma \)-field of all \( Q \)-saturated Borel sets in \( X \). We
shall denote the equivalence relation induced by $Q$ by $R(Q)$. A subset $S$ of $X$ is
called a cross-section of $Q$ if $S \cap E$ is a singleton for all $E \in Q$.

Let $(Z, d)$ be a metric space, $\varepsilon$ a positive real number, $z \in Z$ and $A \subseteq Z$. Then $S^\varepsilon(z)$ will denote the open ball \{ $z' \in Z$ : $d(z, z') < \varepsilon$ \} and $\text{cl}(A)$ will denote the
topological closure of $A$.

We shall now state some lemmas which will be used in the sequel.

**Lemma 2.1.** Let $(T, A)$ be a measurable space, $X$ a Polish space and $F : T \to X$ a
closed valued, $A$-measurable multifunction. If $f : T \to X$ is an $A$-measurable selector
for $F$ and $\varepsilon$ a positive real number then the multifunction $H : T \to X$ defined by

$$H(t) = \text{cl}(F(t) \cap S^\varepsilon(f(t))), \quad t \in T,$$

is $A$-measurable.

A proof of this lemma can be found in [6].

**Lemma 2.2.** Let $T$ be an analytic space and $X$ a Polish space. If $B \in \mathcal{B}_{T \times X}$ and $B^t$ is compact for all $t \in T$ then $\tau_T(B) \in \mathcal{B}_T$.

**Proof.** Embed $T$ in a Polish space $Z$ and let $E \in \mathcal{B}_{Z \times X}$ such that $B = E \cap (T \times X)$. Suppose $P = \{ z \in Z : E^z \text{ is nonempty and compact}\}$. Then $P$ is coanalytic
[5, p. 224] and $\tau_T(B) \subseteq P$. We get a Borel set $R$ in $Z$ such that $\tau_T(B) \subseteq R \subseteq P$ [2, p. 485]. Then $R \cap T = \tau_T(B)$. Hence $\tau_T(B)$ is Borel in $T$.

We shall use the following deep facts about Borel sets with $\sigma$-compact sections in
the sequel.

**Theorem 2.3.** Let $T$ be an analytic and $X$ a Polish space. If $B$ is a Borel subset of
$T \times X$ such that $B^t$ is $\sigma$-compact for every $t \in T$, then

(a) $\tau_T(B)$ is Borel in $T$,

(b) $B$ can be uniformized by a Borel set in $T$, and

(c) there exist Borel sets $B_0, B_1, B_2, \ldots$ in $T \times X$ such that $B = \bigcup_n B_n$ and $B^t_n$ is
compact for all $n$ and $t$.

Part (a) of this theorem was proved independently by Arsenin and Kunugui, (b) is the uniformization theorem of Shchegolkov, and part (c) is a very difficult
theorem of Saint-Raymond [4].

3. Proof of Theorem 1.1. (i) $\Rightarrow$ (ii). Let $A_1, A_2, \ldots$ generate $\mathcal{A}(Q)$ and suppose $f$ is the characteristic function of \{ $A_n$ \}. Then $f$ is $\mathcal{A}(Q)$-measurable and $R(Q) = \{(t, t') \in T \times T : f(t) = f(t')\}$. Therefore, $R(Q) \in \mathcal{A}(Q) \otimes \mathcal{A}(Q) \subseteq \mathcal{A}(Q) \otimes \mathcal{B}_T$.

(ii) $\Rightarrow$ (i). Let $A_1, A_2, \ldots \in \mathcal{A}(Q)$ and $B_1, B_2, \ldots \in \mathcal{B}_T$ be such that $R(Q)$ belongs to the $\sigma$-field generated by $A_1 \times B_1, A_2 \times B_2, \ldots$. Let $M$ be the $\sigma$-field generated by $A_1, A_2, \ldots$. Then $M$ is a countably generated sub-$\sigma$-field of $\mathcal{A}(Q) \subseteq \mathcal{B}_T$ and $M$ and $\mathcal{A}(Q)$ have the same set of atoms. Hence by a theorem of Blackwell
[1, Theorem 3] $M = \mathcal{A}(Q)$. This proves (i).

To prove the last part of the theorem let $A_1, A_2, \ldots$ be a countable generator of
$\mathcal{A}(Q)$ and, suppose $f$ is the characteristic function of the sequence \{ $A_n$ \}. Let
\[ P \subseteq [0, 1] \] be the range of \( f \). Then \( P \) is Borel (Theorem 2.3(a)). Define a multifunction \( H: P \to T \) by

\[ H(p) = \{ t \in T : f(t) = p \}, \quad p \in P. \]

Then \( H(p) \) is \( \sigma \)-compact for all \( p \in P \) and \( \text{graph}(H) = B_{P \times T} \). Therefore, by the uniformization theorem of Shchegolkov, there is a Borel selector \( h: P \to T \) for \( H \).

Put \( S = \{ t \in T : h(f(t)) = t \} \). It is easily seen that \( S \) is a Borel cross-section for \( Q \).

\textbf{Remark.} Consider the partition of the real line \( \mathbb{R} \) given by "\( x \) is equivalent to \( y \) if \( x - y \) is a rational number". Members of this partition are all countable and the saturation of every open set is open. But it is well known that this does not admit even a Lebesgue measurable cross-section. This shows that the hypothesis of the theorem cannot be relaxed.

\textbf{4. Representation theorems.} We first prove a representation theorem for compact valued multifunctions.

\textbf{Lemma 4.1.} If \( (T, \mathcal{A}) \) is a measurable space, \( X \) a compact metric space and \( \varepsilon \) a positive real number then there is a positive integer \( n \) such that for every compact valued, \( \mathcal{A} \)-measurable multifunction \( F: T \to X \) there exist \( \mathcal{A} \)-measurable selectors \( f_0, f_1, \ldots, f_n \) for \( F \) such that \( f_0(t), \ldots, f_n(t) \) is an \( \varepsilon \)-net in \( F(t) \) for all \( t \).

\textbf{Proof.} Let \( n \) be a positive integer that there exist open sets \( W_0, W_1, \ldots, W_n \) in \( X \) of diameters less than \( \varepsilon \) which cover \( X \). Fix a compact valued, \( \mathcal{A} \)-measurable multifunction \( F: T \to X \). Fix an \( \mathcal{A} \)-measurable selector \( g: T \to X \) for \( F \). Define \( T_i = F^{-1}(W_i), 0 < i < n \). Then \( T_i \subseteq \mathcal{A} \) and \( \bigcup_{i=0}^{n} T_i = T \).

For any nonnegative integer \( i \), less than or equal to \( n \), define a multifunction \( F_i: T_i \to W_i \) by

\[ F_i(t) = F(t) \cap W_i, \quad t \in T_i. \]

Then \( F_i \) is \( \mathcal{A} \)-measurable and closed valued. Hence there is an \( \mathcal{A} \)-measurable selector \( g_i: T_i \to W_i \) for \( F_i \) [3]. Now define \( f_i: T \to X \) by

\[ f_i(t) = g_i(t) \quad \text{if} \ t \in T_i, \]

\[ = g(t) \quad \text{if} \ t \in T - T_i. \]

Clearly \( f_0, \ldots, f_n \) have the required properties.

\textbf{Theorem 4.2.} Let \( (T, \mathcal{A}) \) be a measurable space and \( X \) a Polish space. Suppose \( F: T \to X \) is a compact valued, \( \mathcal{A} \)-measurable multifunction. Then there is a point map \( f: T \times C \to X \) such that

(i) for all \( t \in T \), \( f(t, \cdot) \) is a continuous map from \( C \) onto \( F(t) \), and

(ii) for all \( \alpha \in C \), the function \( f(\cdot, \alpha) \) is \( \mathcal{A} \)-measurable.

\textbf{Proof.} Since any Polish space can be embedded in a compact metric space, without any loss of generality, we assume that \( X \) is a compact metric space.

We show that there are positive integers \( n_0, n_1, n_2, \ldots \) and for each \( s \in \text{Seq} \) with \( s_i < n_i, i < \text{lh}(s) \), an \( \mathcal{A} \)-measurable selector \( g_s: T \to X \) for \( F \) such that for all \( t \in T \), \( \{ g_s(t) : i < n_k \} \) is a \( \frac{\varepsilon}{2^{(k+1)\text{lh}(s)}} \)-net in \( \text{cl}(F(t) \cap S_{2^k}(g_s(t))) \) where \( k = \text{lh}(s) \). To show this we proceed by induction.
Let \( g_e \) be any \( A \)-measurable selector for \( F \). By Lemma 4.1 we get a positive integer \( n_0 \) and \( A \)-measurable selectors \( g_0, g_1, \ldots, g_n \) for \( F \) such that \( g_0(t), \ldots, g_n(t) \) is a \( 1/2 \) net in \( F(t) \) for all \( t \). Suppose for some positive integer \( k \), positive integers \( n_0, \ldots, n_{k-1} \) and functions \( g_s \) for \( s \in \text{Seq} \) with \( \text{lh}(s) < k \) and \( s_i < n_i \) for \( 0 < i < \text{lh}(s) \) satisfying the above conditions have been defined. Fix an \( s \in \text{Seq} \) such that \( \text{lh}(s) = k \) and \( s_i < n_i \) for \( 0 < i < k \). Define a multifunction \( F_s: T \rightarrow X \) by

\[
F_s(t) = \text{cl}(F(t) \cap S_{2^{-k}}(g_s(t))), \quad t \in T.
\]

Then by Lemma 2.1, \( F_s \) is \( A \)-measurable. By Lemma 4.1 we get a positive integer \( n_k \) (independent of \( s \)) and \( A \)-measurable selectors \( g_{s0}, g_{s1}, \ldots, g_{s_{n_k}} \) for \( F_s \) such that, for all \( t \), \( g_{s0}(t), \ldots, g_{s_{n_k}}(t) \) is a \( 2^{-(k+1)} \)-net in \( F_s(t) \). It is easy to check that positive integers \( n_0, n_1, \ldots \) and the function \( g_s: T \rightarrow X \) thus defined have desired properties.

Put \( Y = X \times \{0, 1, \ldots, n_1\} \) and give it the product of discrete topologies. Take a \( \delta \in \text{Seq} \) and \( t \in T \). Then \( \{g_{\delta|k}(t): k \in \omega\} \) is a Cauchy sequence in \( X \). Define \( f(t, \delta) = \lim_{k \rightarrow \infty} g_{\delta|k}(t) \). It is easily checked that \( f \) satisfies (i) and (ii). Since \( Y \) is homeomorphic to \( C \) the proof is complete.

**Proof of Theorem 1.2.** Since any Polish space can be embedded in a compact metric space as a \( G_\delta \), without any loss of generality, we assume that \( X \) is a compact metric space.

Let \( G = \text{graph}(F) \) and \( G_0, G_1, G_2, \ldots \) be a sequence of Borel sets in \( T \times X \) such that \( G_n \) is compact for all \( n \) and \( F = \bigcup_n G_n \) (Theorem 2.3(c)).

Fix an \( n \in \omega \). Let \( m_0, m_1, \ldots \) be an enumeration of \( \omega \) such that \( m_0 = n \). Suppose

\[
T_i^n = \pi_T(G_m) \quad \text{if} \quad i = 0,
\]

\[
= \pi_T(G_m) - \bigcup_{j < i} \pi_T(G_m) \quad \text{if} \quad i > 0.
\]

By Lemma 2.2, \( T_i^n \in B_T, i \neq j \rightarrow T_i^n \cap T_j^n = \emptyset \) and \( \bigcup_{i=0}^\infty T_i^n = T \). Define a multifunction \( F_n: T \rightarrow X \) by \( F_n(t) = G_m^n \) if \( t \in T_i^n \). Then \( F_n \) is compact valued and, by Lemma 2.2, \( B_T \)-measurable. By Theorem 4.2 we get a map \( f_n: T \times C \rightarrow X \) such that, for all \( t \in T, f_n(t, \cdot) \) is continuous and onto \( F_n(t) \) and, for all \( \alpha \in C \), the map \( f_n(\cdot, \alpha) \) is \( B_T \)-measurable. Define \( f: T \times (\omega \times C) \rightarrow X \) by

\[
f(t, m, \alpha) = f_m(t, \alpha), \quad t \in T, \ m \in \omega, \ \alpha \in C.
\]

It is easily checked that \( f \) has the desired properties.

**References**


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