COUNTABLE INJECTIVE MODULES ARE SIGMA INJECTIVE

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Abstract. In this note we show that a countable injective module is \( \Sigma \)-injective and consequently a ring \( R \) is left noetherian if the category of left \( R \)-modules has a countable injective cogenerator. Our proof can be modified to establish the corresponding result for quasi-injective modules. We also give an example of a nonnoetherian commutative ring \( R \) such that the category of \( R \)-modules has a countable cogenerator.

We let \( R \) denote an arbitrary ring with identity and \( M \) a unital left \( R \)-module. Recall that \( M \) is injective if and only if for each left ideal \( I \) of \( R \) and each \( R \)-homomorphism \( f : I \to M \) there is a \( y \in M \) such that \( f(r) = ry \) for all \( r \in I \). If \( X \) is a subset of \( M \), then \( l_R(X) \) is the left ideal consisting of those \( r \in R \) such that \( rx = 0 \) for all \( x \in X \). Similarly if \( 7 \) is a subset of \( R \), we let \( r_M(7) = \{ x \in M : \text{I}x = 0 \} \). If an arbitrary direct sum of copies of \( M \) is injective, then \( M \) is said to be \( \Sigma \)-injective. Faith [4] has shown that an injective module \( M \) is \( \Sigma \)-injective if and only if the ascending chain condition holds for the left annihilator ideals \( l_R(X) \).

Theorem. A countable injective module is \( \Sigma \)-injective.

Proof. Let \( y_1, y_2, \ldots, y_n, \ldots \) be an enumeration of the elements of the countable injective \( R \)-module \( M \). Assume by way of contradiction that there exists a strictly ascending chain \( I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \) of left annihilator ideals. If we let \( X_n = r_M(I_n) \), then \( I_n = l_R(X_n) \) and in \( M \) we have the strictly descending chain \( X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots \). Moreover if \( X = \cap_{n=1}^{\infty} X_n \), then \( X = r_M(7) \) where \( 7 = \cup_{n=1}^{\infty} I_n \). We now construct inductively a sequence \( b_1, b_2, \ldots, b_n, \ldots \) in \( I \) and a corresponding sequence of \( R \)-homomorphisms \( f_n : \Sigma_{i=1}^{n} Rb_i \to M \) with \( f_n \subseteq f_{n+1} \) and \( f_n(b_n) \neq b_ny_n \) for all \( n \). For \( n = 1 \), we choose a \( z_1 \in X_1 \) such that \( z_1 - y_1 \notin X \). Since \( X = r_m(7) \) there is some \( b_1 \in I \) such that \( b_1(z_1 - y_1) \neq 0 \) and thus the homomorphism \( f_1 : Rb_1 \to M \) given by right multiplication by \( z_1 \) has the property that \( f_1(b_1) \neq b_1y_1 \). Now suppose we have found \( b_1, \ldots, b_n \) and \( f_1, \ldots, f_n \) with the desired properties. Since \( M \) is injective, there is a \( z_n \) in \( M \) such that \( f_n(r) = rz_n \) for all \( r \) in the domain of \( f_n \). For sufficiently large \( m \), we have \( b_1, \ldots, b_n \) in \( I_m \) and we select \( z_{n+1} \) in \( X_m \) such that \( z_{n+1} + z_n - y_{n+1} \notin X \). Then there will

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exist some \( b_{n+1} \) in \( I \) such that \( b_{n+1}(z_{n+1} + z_n - y_{n+1}) \neq 0 \) and the map \( f_{n+1} : \text{sup}_{i=1}^{\infty} Rb_i \rightarrow M \) given by right multiplication by \( z_{n+1} + z_n \) has the required properties. Finally to obtain the desired contradiction we note that the supremum \( f \) of all the \( f_n \)'s is a homomorphism from the left ideal \( \text{sup}_{i=1}^{\infty} Rb_i \) into \( M \) and therefore there is a \( y \in M \) such that \( f(r) = ry \) for all \( r \) in the domain of \( f \). But this yields \( b_ny = f(b_n) = f_n(b_n) \neq b_ny_n \) for all \( n \), contrary to the fact that \( y \) must equal some \( y_n \).

**Remark.** The foregoing proof is but a slight modification of the argument given by Lawrence [6] to show that a countable self-injective ring is necessarily quasi-Frobenius. As in that paper, this argument can be generalized to show that if \( M \) is an injective \( R \)-module of regular cardinality \( m \), then any well-ordered properly ascending chain in \( R \) of left annihilators of subsets of \( M \) must have length less than \( m \).

Recall that \( M \) is a cogenerator if each left \( R \)-module can be imbedded as a submodule of a product of sufficiently many copies of \( M \). Since it is easily seen that the left ideal \( I \) is the annihilator of a subset of \( M \) if (and only if) \( R/I \) can be imbedded in a product of copies of \( M \), every left ideal of \( R \) will be the annihilator of a subset of \( M \) provided the latter is a cogenerator. Thus we immediately have the following

**Corollary 1.** If the category of left \( R \)-modules has a countable injective cogenerator, then \( R \) is left noetherian.

Let \( J \) be the Jacobson radical of \( R \). We call \( R \) semilocal if \( R/J \) is semisimple. For such a ring \( R \) we have only finitely many isomorphically distinct simple left \( R \)-modules \( S_1, \ldots, S_n \) and as an injective cogenerator we have \( E(S_1) \oplus \cdots \oplus E(S_n) \) where \( E(S) \) is the injective envelope of \( S \). Therefore from Corollary 1 we have the following result.

**Corollary 2.** If \( R \) is semilocal and if the injective envelope of each simple left \( R \)-module is countable, then \( R \) is left noetherian.

Since a nilideal in a left noetherian ring is nilpotent and a semiprimary ring is left artinian if and only if it is left noetherian, we can also make the following observation.

**Corollary 3.** If \( R \) is a semilocal ring with nil-Jacobson radical and if the injective envelope of each simple left \( R \)-module is countable, then \( R \) is left artinian.

Examples exist showing that "injective cogenerator" cannot be weakened to "cogenerator" in Corollary 1 and "semilocal" is an essential hypothesis in corollary 2. Indeed there exist countable, commutative, nonnoetherian rings \( R \) such that for each maximal ideal \( P \) of \( R \) the localization \( R_P \) is a rank one discrete valuation ring. For such a ring \( R \), \( E(S) \) will be countable for each simple \( R \)-module \( S \) (see [7, Theorem 3.11]) in spite of the fact that \( R \) is not noetherian. Moreover as noted in [2] such an \( R \) can be constructed in which exactly one maximal ideal fails to be finitely generated. Under these circumstances \( R \) can contain only countably many maximal ideals which in turn give rise to countably many isomorphically distinct
simple $R$-modules $S_1$, $S_2$, \ldots, $S_n$, \ldots. Then the countable module $M = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n) \oplus \cdots$ is a cogenerator (see, for example, [1, 18.16]), but it is not injective by Corollary 1 since $R$ is not noetherian.

Finally we wish to note that the proof of our theorem can easily be modified to yield the same conclusion for countable quasi-injective modules. Recall that $M$ is quasi-injective if each homomorphism $f: N \to M$ with $N$ a submodule of $M$ extends to an endomorphism of $M$. It is not difficult to generalize a result of Fuchs [5] in order to show that $M$ is quasi-injective if and only if it satisfies the following condition: If $I$ is a left ideal of $R$ and if $f: I \to M$ is an $R$-homomorphism with $\ker f \supseteq 1_R(F)$ for some finite subset $F$ of $M$, then there is a $y \in M$ such that $f(r) = ry$ for all $r \in I$. Then armed with the characterization of $\Sigma$-quasi-injective modules given in [3], one can readily carry out the desired proof that countable quasi-injective modules are $\Sigma$-quasi-injective.

REFERENCES


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