ON ANISOTROPIC SOLVABLE LINEAR ALGEBRAIC GROUPS

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Abstract. A connected linear algebraic solvable group $G$ defined over a field $k$ is anisotropic over $k$ if $G$ has no $k$-subgroup splitting over $k$. A simple criterion for anisotropic solvable groups is presented when $k$ is a local field.

Let $G$ be a connected linear algebraic solvable group defined over a field $k$. The group $G$ is said to be splitting over $k$ if $G$ has a normal series of $k$-subgroups such that the factor groups are $k$-isomorphic either to the additive group $G_a$ or the multiplicative group $G_m$. We say that $G$ is anisotropic over $k$ if $G$ has no $k$-subgroups splitting over $k$. In this note, we give a criterion for anisotropic solvable groups in terms of compactness condition when $k$ is a local field. Our main result is the following theorem.

Theorem M. Let $G$ be a connected linear algebraic solvable group defined over a local field $k$. Then the following conditions are equivalent.

(i) $G$ is anisotropic over $k$.

(ii) $G$ is nilpotent, and both the maximal torus $T$ of $G$ and the quotient group $G/T$ are anisotropic over $k$.

(iii) The group $G(k)$ of $k$-rational points of $G$ is compact where $G(k)$ is endowed with the locally compact topology from that of $k$.

When $G$ is a torus, the result is well known. The argument of the next lemma is due to Prasad [2].

Lemma 1. Let $T$ be a torus defined over a local field $k$. Then $T(k)$ is compact if and only if $T$ is anisotropic over $k$.

Proof. We know that $T$ is splitting over a finite Galois extension $K$ of $k$. Clearly, $T(k)$ is a closed subgroup of $T(K)$. From this $T(k)$ is compact if and only if for every $t \in T(k)$ and character $\chi$ of $T$, $\chi(t)$ is of absolute value 1. If $T(k)$ is not compact, then there exists $t \in T(k)$ such that for at least one character $\chi$ of $T$, $\chi(t)$ has absolute value $\neq 1$. This implies that $\Sigma_{\sigma \in \text{Gal}(K/k)} \sigma_{\chi(t)}$ also has absolute value $\neq 1$. Thus the character $\Sigma_{\sigma \in \text{Gal}(K/k)} \sigma_{\chi}$ is nontrivial and defined over $k$. This shows that $T$ is $k$-isotropic.

For unipotent groups, we need more lemmas.
Lemma 2. Let $k$ be a local field with characteristic $\text{ch}(k) = p > 0$ and $A$ a subset of $k^n$. If $f$ is an additive $k$-morphism of $G_a^n$ such that $f(A)$ is relatively compact in $k$, then up to a $k$-automorphism of $G_a^n$, there exists an integer $r$ with $0 < r < n$ satisfying the following conditions.

(i) $f$ is independent of the first $r$ coordinates.
(ii) Let $p_r$ be the projection of $G_a^n$ onto the last $n - r$ coordinates. The projection $p_r(A)$ of $A$ is relatively compact in $k^{n-r}$.

Proof. Clearly, we may assume that $f$ is nontrivial. For $1 < i < n$, we define an additive $k$-morphism $f_i$ of $G_a$ by $f_i = f \circ \iota_i$ where $\iota_i$ is the inclusion map of $G_a$ into the $i$th component. Since $f$ is additive, for $x = (x_1, \ldots, x_n) \in G_a^n$, we have

$$f(x) = f_1(x_1) + \cdots + f_n(x_n).$$

Denote by $I$ the set of indices $j$ with $f_j \neq 0$. After replacing $f$ by $f \circ \alpha$ where $\alpha$ is a $k$-automorphism of $G_a^n$, we may assume that the cardinality of $I$ is minimal. Hence it suffices to show that $A$ is relatively compact when $I = \{1, 2, \ldots, n\}$. Suppose that the assertion is false. There exists a sequence $\xi_m = (\xi_1(m), \ldots, \xi_n(m))$ of elements in $A$ such that the norms $||\xi_m||$ ($m = 1, 2, \ldots$) are not bounded. The maps $f_i$ ($i = 1, \ldots, n$) are additive $k$-morphisms of $G_a$. Hence we can write

$$f_i(t) = a_{i,0}t + a_{i,1}t^p + \cdots + a_{i,p}t^{p^r},$$

with $a_{i,p} \neq 0$ ($i = 1, \ldots, n$). Here we may assume that the number $\Sigma I_s$ has been chosen to be minimal. After replacing $(\xi_m)$ by a subsequence and up to a $k$-automorphism of $G_a^n$, there is a positive integer $v < n$ satisfying the following conditions.

(1) $\xi_i(m) \rightarrow \infty$, \quad $1 \leq i \leq v$.
(2) For $i, j < v$, the numbers $p^s \ord_k(\xi_i(m)) - p^s \ord_k(\xi_j(m))$ are independent of $m$.
(2.1) For $i < v, j > v$, the sequence $p^s \ord_k(\xi_i(m)) - p^s \ord_k(\xi_j(m))$ tends to $\infty$.

Now let $s = \max\{s_1, \ldots, s_p\}$ and assume, as we may, that $s = s_1$. Since $f(A)$ is relatively compact in $k$, by (1) of (2.1), the sequence $f(\xi_m)\xi_i(m)^{-p^s}$ converges to zero, and by (2) and (3) of (2.1) the sequence $b_m$,

$$b_m = a_{1,s_1} + a_{2,s_2}(\xi_2(m)\xi_2(m)^{-p^{s_1-1}})^{p^s} + \cdots + a_{v,s_v}(\xi_v(m)\xi_v(m)^{-p^{s_1-1}})^{p^s},$$

converges to zero. It follows readily from (2) of (2.1) that there exist $\xi_2, \ldots, \xi_v \in k$ such that

$$a_{1,s_1} + a_{2,s_2}\xi_2^{p^s} + \cdots + a_{v,s_v}\xi_v^{p^s} = 0.$$ 

Then we have the identity

$$a_{1,s_1}x_1^{p^s} + \cdots + a_{v,s_v}x_v^{p^s}$$

$$= a_{2,s_2}(x_2 - \xi_2x_1^{p^{s_1-1}})^{p^s} + \cdots + a_{v,s_v}(x_v - \xi_vx_1^{p^{s_1-1}})^{p^s}.$$
Thus if we set $x'_j = x_j - \xi_j x'^{n-1}_j (j = 2, \ldots, \nu)$ and $x'_i = x_i, i \notin \{2, \ldots, \nu\}$, it is easy to verify that in the coordinates $(x'_1, \ldots, x'_n)$
\[
\deg(f_i(x'_i)) < \deg(f_i(x_i))
\]
and
\[
\deg(f_i(x'_i)) = \deg(f_i(x_i)), \quad (1 < i < n),
\]
where $\deg$ is the degree of a polynomial. Obviously we arrive at a contradiction to our choice of minimality of $\Sigma_{i=1}^{a-1} s_i$. Therefore $A$ has to be relatively compact in $k^n$ and the lemma is proved.

**Lemma 3.** Let $k$ be as in Lemma 2, $A$ a subset of $k^n$ and $f_1, \ldots, f_l$ additive $k$-morphisms of $G_a^n$. Suppose that the images $f_j(a)$ are relatively compact in $k$ ($i = 1, \ldots, l$). Then $G_a^n$ has a decomposition $G_a^n = H \times L$ defined over $k$ such that $H \simeq G_a^{t}$, $L \simeq G_a^{n-t}$ over $k$. $H \subset \ker(f_j) (j = 1, \ldots, l)$ and $pr_L(A)$ is relatively compact in $L(k)$ where $pr_L$ is the projection map of $G_a^n$ into $L$.

**Proof.** We may assume that $A$ is not relatively compact in $k^n$. By Lemma 2, $G_a^n$ has a decomposition $G_a^n = M \times N$ defined over $k$ such that $M \simeq G_a^t$, $N \simeq G_a^{n-t}$ over $k$, $t > 0$, and $M \subset \ker(f_j)$, and the projection $pr_N(A)$ of $A$ in $N$ is relatively compact in $N(k)$. Now let $A_1 = pr_M(A)$. Clearly $A_1, f_2|_M, \ldots, f_l|_M$ satisfy all the conditions in Lemma 3. By induction on $l$, our assertion is true in $M$ and consequently in $G_a^n$.

**Proposition 4.** Let $k$ be a local field and $G$ a $k$-subgroup of $G_a^n$. Then $G_a^n$ has a decomposition $G_a^n = H \times L$ defined over $k$ such that $H \simeq G_a^t$, $L \simeq G_a^{n-t}$ over $k$, $H \subset G$ and $(G \cap L)(k)$ is compact.

**Proof.** We may assume that $\text{ch}(k) = p > 0$. By [4, p. 102, Proposition], there exist additive $k$-morphisms $f_1, \ldots, f_l$ such that $G = \cap_{i=1}^l \ker(f_i)$. Now the proposition is an immediate consequence of Lemma 3.

**Theorem 5.** Let $G$ be a connected linear algebraic unipotent group defined over a local field $k$. The following conditions are equivalent.

(i) $G$ is anisotropic over $k$.

(ii) There exist no nontrivial additive $k$-morphisms from $G_a$ into $G$.

(iii) $(G(k))$ is compact.

**Proof.** If $\text{ch}(k) = 0$, $G$ is always $k$-splitting. In this case, all three conditions are equivalent to $G = \{1\}$. Hence we may assume that $\text{ch}(k) = p > 0$ and prove the theorem in several steps.

Clearly, (iii) \(\rightarrow\) (i) \(\rightarrow\) (ii). Thus we show (ii) \(\rightarrow\) (iii). Condition (ii) is equivalent to the condition that the maximal $k$-splitting subgroup of $G$ is $\{1\}$.

**Step 1.** $G$ is commutative and $G^p = \{1\}$. We know [3, p. 34, Corollary 2] that $G$ is isomorphic to $G_a^m$ over $k^{p^m}$ for certain nonnegative integers $m, l$. Hence there is an isomorphism $G \rightarrow G_a^m$ defined over $k^{p^m}$. Let $f: G \rightarrow G_a^m$ be the $k$-morphism given by $f(x) = \tau(x)^{p^m} (x \in G)$. Clearly, $\ker(f) = \{1\}$. Express $\tau$ in the form $\tau = \sum_{i=1}^r \omega_i \tau_i$ where $\tau_i$ are defined over $k$ and $\omega_i (\in k^{p^r})$ are linearly independent.
over $k$. It is easy to see that for $x, y \in G(k)$, $\tau_a(x + y) = \tau_a(x) + \tau_a(y)$. Since $G(k)$ is Zariski-dense in $G$, the maps $\tau_a$ are $k$-morphisms of $G$ into $G^m_a$. By assumption on $\tau$, the differential $d\tau$ of $\tau$ is an isomorphism, it follows readily that $\cap_a \ker(d\tau_a) = \{0\}$. Therefore the map $g: G \to G^m_a$ given by $g(x) = (\tau_a(x)) (x \in G)$ is a separable $k$-morphism. Now using $f$ and $g$, we define $\omega: G \to G^{|\tau| + 1}_a$ by $\omega(x) = (f(x), g(x)) (x \in G)$. Clearly, $\omega$ defines a $k$-embedding of $G$ into $G^{|\tau| + 1}_a$. From Proposition 4, $G(k)$ has to be compact.

**Step 2.** Suppose that $G$ has a connected normal $k$-subgroup $N$ with $\{1\} \neq N \neq G$. Let $L = G/N$, and $L'$ its maximal $k$-splitting subgroup. If $L' \neq L$, let $H$ be the inverse image of $L'$ in $G$. By induction on dimension, $H(k)$ and $(G/H)(k)$ are compact. Since the image of $G(k)$ in $(G/H)(k)$ is open, it follows that $G(k)/H(k)$ is compact, thus so is $G(k)$.

**Step 3.** $G$ is commutative and $G' \neq \{1\}$. Let $l$ be the largest integer with $G^{l+1} \neq \{1\}$ and $N = G^{l'}$. Let $L = G/N$ and $L'$ the maximal $k$-splitting subgroup of $L$. If $L \neq L'$, by Step 2, $G(k)$ is compact. If $L = L'$, the map $x \mapsto x^l (x \in G)$ factors through $L$. Then $G^l$, as a homomorphic image of a $k$-splitting unipotent group, by [3, p. 35, Proposition 6] is $k$-splitting. However, $G^l \neq \{1\}$ and by condition (ii), this is impossible.

**Step 4.** $G$ is not commutative. Let $N = [G, G]$, $L = G/N$ and $L'$ the maximal $k$-splitting subgroup of $L$. Suppose that $L = L'$. Let $H$ be the last term in the lower central series with $H \subseteq Z(G)$ where $Z(G)$ is the center of $G$. Then choose any $h \in H(k)$ such that $h \notin Z(G)$ and consider the map $x \mapsto xhx^{-1}h^{-1} (x \in G)$. The image of the map is in $Z(G)$ by our choice of $H$, hence is a $k$-morphism of algebraic groups. It factors through $L$. Therefore $[h, G]$, by [3, Proposition 6] is $k$-splitting. By (ii), $[h, G]$ is anisotropic over $k$, thus $[h, G] = \{1\}$. However $h \notin Z(G)$, we have a contradiction. Therefore $L' \neq L$ and by Step 2, $G(k)$ is compact.

Now are ready to prove our main result.

**Proof.** When $\text{ch}(k) = 0$, all the three conditions are equivalent to that $G$ is an isotropic torus for $R_u(G)$ is always splitting over $k$. Hence we may assume that $\text{ch}(k) = p > 0$.

(i) $\rightarrow$ (ii). By [4, p. 114, Corollary 2], $G$ is nilpotent. Clearly, $T$ is anisotropic over $k$. Let $H$ be the maximal $k$-splitting subgroup of $G/T$ and $L$ its preimage in $G$. Since $T$ is splitting over a finite separable extension $K$ of $k$, $L$ is splitting over $K$. This implies that $R_u(L)$ is defined over $K$. On the other hand, $L$ is defined over $k$, so $R_u(L)$ is $k$-closed. Thus $R_u(L)$ is defined over $k$. As $R_u(L)$ is $k$-isomorphic to $L/T = H$, $R_u(L)$ is splitting over $k$. Therefore $R_u(L) = \{1\}$ and so is $H = \{1\}$.

(ii) $\rightarrow$ (iii). From Lemma 1 and Theorem 5, $T(k)$ and $(G/T)(k)$ are compact. We know that the image of $G(k)$ in $(G/T)(k)$ is open, hence compact. It follows readily that $G(k)$ is compact because $T(k)$ and $G(k)/T(k)$ are compact.

(iii) $\rightarrow$ (i) is obvious.
REFERENCES


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