SUBHARMONIC FUNCTIONS OUTSIDE A COMPACT SET IN $\mathbb{R}^n$

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Abstract. Let $u$ be a subharmonic function defined outside a compact set in $\mathbb{R}^2$. Then $u$ is of the form $u(x) = s(x) - \alpha \log|x|$ outside a disc where $s(x)$ is a nonconstant subharmonic function in $\mathbb{R}^2$ and $\alpha > 0$. Some applications and the analogues in $\mathbb{R}^n$, $n > 3$, are given.

1. Introduction. Let $u$ be a subharmonic function defined outside a compact set in $\mathbb{R}^2$. We prove that outside a disc $u$ is of the form $u(x) = s(x) - \alpha \log|x|$ where $s(x)$ is a nonconstant subharmonic function in $\mathbb{R}^2$ and $\alpha > 0$.

From the above decomposition, it is easily seen that $\mu(u) = \lim(M(r, u)/\log r)$ always exists where $M(r, u)$ is the mean value of $u$ on $|x| = r$. We show that there exists a (nonharmonic) subharmonic function $v$ in $\mathbb{R}^2$ such that $v = u$ outside a disc if and only if $\mu(u) > 0$. This result is implicit in M. Brelot [1] and it is proved here as a simple application of the above decomposition of $u$.

In particular, defining the order of $u$ as $\text{ord } u = \text{ord } s$, we note that there exists a (nonharmonic) subharmonic function $v$ in $\mathbb{R}^2$ such that $v = u$ outside a disc in the following two cases: (i) $u$ is of nonintegral order, and (ii) $u$ is lower bounded but not bounded.

Finally we state some of the analogous results in $\mathbb{R}^n$, $n > 3$.

2. Subharmonic functions outside a disc in $\mathbb{R}^2$.

Theorem 1. Suppose that $u$ is a subharmonic function defined outside a compact set in $\mathbb{R}^2$. Then there exist a nonconstant subharmonic function $s$ in $\mathbb{R}^2$ and a constant $\alpha > 0$ such that $u(x) = s(x) - \alpha \log|x|$ outside a disc.

Proof. Let $u$ be finite continuous in a neighbourhood of $|x| = R > 1$ and subharmonic in $|x| > R_0$ ($R_0 < R$). Let $r > R$ and $D_r u$ denote the Dirichlet solution in $|x| < r$ with boundary value $u$.

Choose $\alpha > 0$ large so that $D_r u + \alpha \log r > u + \alpha \log R$ on $|x| = R$. This implies that $u(x) + \alpha \log|x| < D_r(u + \alpha \log|x|)$ in $R < |x| < r$.

Hence if $s(x)$ is the function $u(x) + \alpha \log|x|$ in $|x| > r$ extended by $D_r(u + \alpha \log|x|)$ in $|x| < r$, $s(x)$ is subharmonic in $\mathbb{R}^2$ and $u(x) = s(x) - \alpha \log|x|$ outside a disc. Moreover $s$ can always be chosen as a nonconstant function since $\alpha$ is an arbitrary large positive number.

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Remarks. (1) Since \( \lim(M(r, s)/\log r) \) exists and equals the total mass associated with \( s \) in the local Riesz representation, \( \mu(u) = \lim(M(r, u)/\log r) \) always exists and \(-\infty < \mu(u) < \infty\).

(2) When \( \mu(u) < \infty \), the least harmonic majorant of \( u \) outside a disc is of the form \( \mu(u) \log|x| + a \) harmonic function \( H(x) \) in \( \mathbb{R}^2 \) + a bounded harmonic function \( b(x) \).

(3) With the representation of \( u \) as in Theorem 1, we define the order of \( u \) as \( \text{ord } u = \text{ord } s \). (For the definition of the order of a subharmonic function in \( \mathbb{R}^n \), \( n > 2 \), we refer for instance to p. 143 of W. K. Hayman and P. B. Kennedy [2].) Note that \( \text{ord } u \) is independent of the representation. For, if \( u = s_1 - \alpha \log|x| \) is another representation of \( u \) outside a disc, then \( \text{ord } s = \text{ord } (s + \alpha \log|x|) = \text{ord } (s_1 + \alpha \log|x|) = \text{ord } s_1 \).

3. Subharmonic extension in \( \mathbb{R}^2 \).

Terminology. In this section we denote by \( u \) a subharmonic function defined outside a compact set in \( \mathbb{R}^2 \). We say that a subharmonic function \( V \) in \( \mathbb{R}^2 \) extends \( u \) if \( V = u \) outside a disc.

Now a part of Théorème 2 of M. Brelot [1] can be stated as follows:

Let \( u \) be a subharmonic function defined outside a disc in \( \mathbb{R}^2 \). Then \( V = \lim D_r u \) exists locally uniformly, and \( V \equiv \infty \) if and only if the flux at infinity of \( u \) is \( > 0 \).

As a consequence, for sufficiently large \( r \), \( D_r u \) extended by \( u \) is a (nonharmonic) subharmonic function in \( \mathbb{R}^2 \) if and only if flux \( u \) \( > 0 \). Essentially the same result is obtained below as a simple application of Théorèm 1.

**Theorem 2.** Given \( u \) there exists a (nonharmonic) subharmonic function in \( \mathbb{R}^2 \) extending \( u \) if and only if \( \mu(u) > 0 \).

**Proof.** If there is a function \( q \) (nonharmonic) subharmonic in \( \mathbb{R}^2 \) such that \( q = u \) outside a disc, then clearly \( \mu(u) = \lim(M(r, q)/\log r) > 0 \) since \( q \) is not harmonic.

Conversely, let \( u(x) = s(x) - \alpha \log|x| \) be a representation of \( u \). Let \( \lambda \) be the measure associated with \( s \). Since \( \mu(u) > 0 \), \( ||\lambda|| > \alpha \). Choose a compact \( K \) such that \( \lambda(K) > \alpha \).

Write \( s = s_1 + s_2 \) where \( s_2 \) is the logarithmic potential corresponding to \( \lambda \) restricted to \( K \) and \( s_1 \) is a subharmonic function in \( \mathbb{R}^2 \) with associated measure \( \lambda \) restricted to \( \mathbb{R}^2 - K \).

Since \( s_2(x) - \lambda(K) \log|x| \to 0 \) as \( |x| \to \infty \), \( D_r(s_2 - \alpha \log|x|) \to \infty \) locally uniformly as \( r \to \infty \). Consequently \( D_r u \to \infty \) locally uniformly.

Hence, for large \( r \), \( D_r u > u \) on \( |x| = R \) which implies that \( D_r u > u \) in \( R < |x| < r \). Define the function \( q \) as \( D_r u \) in \( |x| < r \) extended by \( u \) in \( |x| > r \). Then \( q \) is a subharmonic function in \( \mathbb{R}^2 \) extending \( u \).

**Corollary 1.** Let \( u \) be lower bounded but not bounded in \( |x| > r \). Then there exists a (nonharmonic) subharmonic function in \( \mathbb{R}^2 \) extending \( u \).

**Proof.** Let \( u > m \) outside a disc. Then \( \mu(u) > 0 \). We show now that \( \mu(u) > 0 \) and hence the corollary follows from the above theorem.
For that, suppose \( \mu(u) = 0 \). Then if \( h \) is the least harmonic majorant of \( u \) outside a disc, \( h \) is of the form \( h(x) = a \) harmonic function \( H(x) \) in \( \mathbb{R}^2 \) + a bounded harmonic function \( b(x) \). (See Remark 2 above.)

This implies that \( m < u(x) < H(x) + b(x) \) outside a disc and hence \( H \) is a constant, which in turn implies that \( u \) is bounded, a contradiction.

**Corollary 2.** Let \( u \) be of finite nonintegral order. Then there exists a (non-harmonic) subharmonic function in \( \mathbb{R}^2 \) extending \( u \).

**Proof.** In this case we show that \( \mu(u) = \infty \) and hence the corollary follows from Theorem 2.

For this, suppose that \( \mu(u) < \infty \). Then the least harmonic majorant of \( u \) outside a disc is of the form \( \mu(u) \log|x| + a \) harmonic \( H(x) \) in \( \mathbb{R}^2 \) + a bounded harmonic function \( b(x) \).

Let \( u(x) = s(x) - \alpha \log|x| \) be a decomposition of \( u \). Then from \( s - \alpha \log|x| < \mu(u) \log|x| + H + b \) outside a disc, it follows that \( \text{ord } s = \text{ord } H \).

Since \( \text{ord } u \) (= \( \text{ord } s \)) is finite and \( H \) is harmonic in \( \mathbb{R}^2 \), it now follows (for instance from Theorem 2.1.5 of W. K. Hayman and P. B. Kennedy [2]) that \( \text{ord } u \) is an integer, a contradiction.

4. In higher dimensions. We state here two theorems in \( \mathbb{R}^n \), \( n > 3 \), analogous to those proved earlier in \( \mathbb{R}^2 \).

**Theorem 1'.** Let \( u \) be a subharmonic function defined in \( |x| > R \) in \( \mathbb{R}^n \), \( n > 3 \). Let \( r > R \). Then there exist a nonconstant subharmonic function \( s(x) \) in \( \mathbb{R}^n \) and a constant \( \alpha < 0 \) such that \( u(x) = s(x) - \alpha |x|^{2-n} \) in \( |x| > r \).

**Theorem 2'.** Let \( u \) be a subharmonic function defined in \( |x| > R \) in \( \mathbb{R}^n \), \( n > 3 \), with associated measure \( \mu \). Let \( r > R \). Then the following are equivalent:

(i) \( \lim M(r, u) = \infty \).

(ii) \( \int_0^\infty |y|^{2-n} \, d\mu(y) \) is divergent.

(iii) There exists a subharmonic function \( v \) in \( \mathbb{R}^n \), not majorized by any harmonic function, extending \( u \).

**Bibliography**


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