AN EXTREMAL VECTOR-VALUED $L^p$-FUNCTION TAKING NO EXTREMAL VECTORS AS VALUES

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Abstract. We give an example of a nonseparable Banach space $V$ and a function $x$ on $[0, 1]$ with values in the unit sphere of $V$ that is an extreme point of the unit balls of all Bochner $L^p$-spaces $L^p(\lambda, V)$, $1 < p < \infty$, Lebesgue measure, though none of its values is an extreme point of the unit ball of $V$. This shows that a characterization of the extremal elements in $L^p(\lambda, V)$ for separable $V$, given by J. A. Johnson, does not hold in general.

The extremal elements in the unit sphere of vector-valued $CK$- or $L^p$-spaces have been studied by many authors, e.g. in [2, 5 and 6]. (For the definition and elementary properties of Bochner $L^p$-spaces we refer the reader to [4].) A quite natural question is to ask whether such a function $x$ is extremal if and only if

1. the function $\|x(\cdot)\|$ is extremal in the corresponding scalar function space and

2. the vector $x(t)$ is extremal in the ball with radius $\|x(t)\|$ for all $t$ in a dense subset of the base space (in the $CK$-case) resp. for almost all $t$ (in the $L^p$-case).

The "if" part is easy and well known; on the other hand, necessity of (1) is trivial. Hence the remaining question is the necessity of (2).

In the case $p = 1$ the necessity is easily seen, since (1) implies that the support of $x$ is an atom (see also [6]).

The $CK$-case was settled long ago. Blumenthal, Lindenstrauss, and Phelps have shown in [2] that for real range spaces $V$ with dimension $< 3$ the condition (2) is necessary. On the other hand they give an example of a 4-dimensional space $V$ and an extremal $x$ in $C([0, 1], V)$ taking no extremal values. In the remaining cases $1 < p < \infty$, J. A. Johnson [5] has shown the necessity of (2), provided $V$ is separable and the measure is a Borel measure on a Polish space (see also [6]).

We shall give an example of a (nonseparable) Banach space $W$ and a function $f: [0, 1] \to W$ that is extremal in the unit balls of all $L^p(\lambda, W)$ ($1 < p < \infty$, $\lambda$ Lebesgue measure), although the function does not take any extremal values.

We start from a Banach space $W$ and a function $f: [0, 1] \to W$, extremal in $C([0, 1], W)$, but taking no extremal values, that came up in a discussion with E. Behrends and R. Evans. Then we show that this example works also in the cases $L^p(\lambda, W)$, $1 < p < \infty$, using the representation $L^p(\lambda, W) \cong L^p(m, W)$ where $m$ is a suitable measure on some Stonean space.

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Let $K$ be the Stonean space of the measure algebra $B/\lambda$ (Borel sets modulo $\lambda$-null sets) and $m$ be the perfect Borel measure on $K$ uniquely determined by $m(C) = \lambda(M)$, where the clopen subset $C$ of $K$ represents the equivalence class of $M$. (Here $m$ is called perfect if each open set has positive measure and each nowhere dense set has measure zero. Each Borel set is the symmetric difference of a clopen and a first category—in fact, nowhere dense—set. See [1], e.g., for details.)

Then the Stone representation can be extended (via simple functions) to an isometry $T: L^p(\lambda, V) \rightarrow L^p(m, V)$. (In the case $p = \infty$, where in general the simple functions are not dense, look at the dense subspace of functions taking on countably many values.)

In the scalar case each equivalence class $x$ in $L^\infty(m)$ contains exactly one continuous function; hence $L^\infty(m) = C(K)$. Thus $K$ is the maximal ideal space of the algebra $L^\infty[0, 1]$. The adjoint of the embedding $C[0, 1] \rightarrow L^\infty[0, 1]$, restricted to $K$, is a continuous, hence Borel measurable, surjection $\omega: K \rightarrow [0, 1]$. Looking at the system of closed intervals with nonvoid interior, which generates the Borel algebra, $\omega$ is easily seen to be inverse measure preserving and to induce the Stone representation $\Psi$ in the sense that $\omega^{-1}(M)$ is equivalent to $\Psi(M)$ for each Borel set $M$.

It follows that $f \mapsto f \circ \omega$ is the isometry $T$ mentioned above. Now let us give the example for the $C(K)$-case.

1. Example. Let $W_0$ be a 3-dimensional space such that there is a curve $f_0: [0, 1] \rightarrow B_0$, $B_0$ the unit ball of $W_0$, with

$$f_0([0, 1]) \subset \text{ex } B_0 \quad \text{and} \quad f_0(0) \notin \text{ex } B_0.$$

(E.g., let $B_0 \subset \mathbb{R}^3$ be the convex hull of $\{(x_1, x_2, 0) | \max |x_i| < 1 \} \cup \{(0, x_2, x_3) | x_2^2 + x_3^2 = 1 \}$ and $f_0(t) := (0, \cos(\pi t/2), \sin(\pi t/2))$.) Then define

$$W := \prod_{[0,1]}^\infty W_0,$$

an $l^\infty$-product of uncountably many copies of $W_0$, $f: [0, 1] \rightarrow W$ by

$$f(s)(t) := f_0(|s + t - 1|)$$

and $x: K \rightarrow W$ by $x := f \circ \omega$. Evidently $x$ is continuous, and $\|x(k)\| = 1$ for all $k$ in $K$. For no $k$ in $K$ is $x(k)$ extremal, because $x(k)(1 - \omega(k)) = f_0(0)$ is not extremal in $B_0$.

However, $x$ itself is extremal. Assume $x = \frac{1}{2}(y + z)$, $y$ and $z$ in the unit ball of $C(K, W)$. Let $k \in K$. We have to show $x(k) = y(k)$, i.e. $x(k)(t) = y(k)(t)$ for all $t$ in $[0, 1]$. This holds for $t \neq 1 - \omega(k)$, as in this case $x(k)(t)$ is extremal in $B_0$.

For $t = 1 - \omega(k)$ we choose a net $(k_\alpha)$ in $K$ converging to $k$, with $\omega(k_\alpha) \neq \omega(k)$ for all indices $\alpha$. This is possible, because $\omega^{-1}(\{\omega(k)\})$ is a closed set of measure zero, and hence has void interior. Then we have $t \neq 1 - \omega(k_\alpha)$, and so $y(k_\alpha)(t) = x(k_\alpha)(t) \rightarrow x(k)(t)$, which in turn yields $x(k)(t) = y(k)(t)$.

\[\square\]
2. **Theorem.** Let $K$ and $m$ be as above, $V$ a Banach space, $x: K 	o V$ and $1 < p < \infty$. Then for the following conditions

(i) $x$ extremal in $C(K, V)$,
(ii) $x$ extremal in $L^\infty(m, V)$,
(iii) $x$ extremal in $L^p(m, V)$ we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

**Proof.** (ii) $\Rightarrow$ (iii) is essentially contained in Theorem 1 in [6]. Assume $x$ is extremal in the unit ball of $L^\infty(m, V)$, in particular $\|x(\cdot)\| = 1$ almost everywhere, and $x = \frac{1}{2}(y + z)$ with $y$ and $z$ in the unit ball of $L^p(m, V)$. Then the Clarkson inequalities [3], applied to the functions $\|y(\cdot)\|$ and $\|z(\cdot)\|$ in $L^p(m)$, yield $\|y(\cdot)\| = \|z(\cdot)\| = \|x(\cdot)\|$ almost everywhere; hence $y$ and $z$ are in the unit ball of $L^\infty(m, V)$.

For (i) $\Rightarrow$ (ii) assume $x$ is extremal in the unit ball of $C(K, V)$ and $x = \frac{1}{2}(y + z)$ with $y$ and $z$ in the unit ball of $L^\infty(m, V)$. We look at $y$ and $z$ as functions rather than equivalence classes, in such a way that the equality $x(k) = \frac{1}{2}(y(k) + z(k))$ holds everywhere (without loss of generality). Now an iterative application of Egorov's theorem and of the regularity of $m$ shows that $y$, as an $m$-almost uniform limit of continuous simple functions, is continuous on a (disjoint) union $U = \bigcup_{n \in \mathbb{N}} C_n$, with $C_n$ clopen and $\sum_{n \in \mathbb{N}} m(C_n) = 1$. That means $y$ and $z$ are continuous on the open complement $U$ of a suitable $m$-null set. For each clopen subset $C$ of $U$, $x|_C$ is extremal in $C(C, V)$; hence $x|_C = y|_C$. Thus $x|_U = y|_U$, which means $x = y$ in $L^\infty(m, V)$. □

3. **Corollary.** The function $f: [0, 1] \to W$ in Example 1, taking no extremal vectors as values, is extremal in the unit ball of $L^p(\lambda, W)$ for $1 < p < \infty$.

**Proof.** The mapping $f \mapsto f \circ \omega$ is a linear isometry of $L^p(\lambda, W)$ onto $L^p(m, W)$. □

4. **Remarks.** (a) An essential point in Theorem 2 was the fact that each Bochner measurable function is continuous on a suitable open set with a null set as complement. So it is not surprising that this theorem fails for $\lambda$ instead of $m$: combine Johnson's result with the 4-dimensional Blumenthal-Lindenstrauss-Phelps example.

(b) The construction of the isometry preceding Example 1 applies to arbitrary finite (even infinite) measures $\mu$,

$$T: L^p(\mu, V) \cong L^p(m, V),$$

and the proof of Theorem 2 holds for these measures $m, V$, too. It would be interesting to know whether in general a Banach space $V$ "recognizes" the extremal elements in $L^p(\mu, V)$ (i.e., extremal functions $x$ satisfy condition (2) above) if and only if it recognizes those in $L^p(m, V)$. Thus, if the above isometry $T$ is not induced by a point mapping $\omega$ as in the case $\mu = \lambda$, is it still true that the ranges of $x$ and $Tx$ are essentially the same?
BIBLIOGRAPHY


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