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TWO UC-SETS WHOSE UNION IS NOT A UC-SET

JOHN J. F. FOURNIER[1]

ABSTRACT. It is shown that the union of two sets of uniform convergence need not be a set of uniform convergence.

We use the standard terminology of harmonic analysis on the unit circle as in [4]. We recall some notions discussed in [8] and [9], and in the references cited in these papers.

DEFINITION. Given a subset E of the integers, call an integrable function f, on the circle, an *E-function* if $\hat{f}(n) = 0$ for all integers n outside E, and denote the space of continuous E-functions by C_E. Call E a *set of uniform convergence*, or *a UC-set*, if every function in C_E has a uniformly convergent Fourier series.

The union problem for UC-sets is mentioned as an open problem in [5, p. 86; 9, p. 283]. To solve it, we need a few more facts about UC-sets. It is known that E is a UC-set if and only if there is a constant κ so that, for each function f in C_E, the partial sums $S_N(f)$ of the Fourier series of f satisfy the inequality $\|S_N(f)\|_\infty \leqslant \kappa\|f\|_\infty$ for all nonnegative integers N. Furthermore, when E is a UC-set, there is a smallest value of κ for which the inequality above holds for all such f and N; this smallest value of κ is called the *UC-constant* of E, and is denoted by $\kappa(E)$. If E is a UC-set, then so is every translate of E, but it turns out that the translates of a UC-set do not all have to have the same UC-constant.

DEFINITION. Call E a *CUC-set*, or a *set of completely uniform convergence* if E is a UC-set with the additional property that the sequence $\{\kappa(E + n)\}_{n=-\infty}^{\infty}$ is bounded.

This notion was introduced, independently by G. Travaglini [9, Lemma 6] and F. Ricci [7, p. 426]. In [8], P. M. Soardi and Travaglini gave some nontrivial examples of CUC-sets, and they showed that if there is a UC-set that is not a CUC-set, then there is a pair of UC-sets whose union is not a UC-set. In the present paper, we exhibit a class of UC-sets that are not CUC-sets, thereby showing that the union of two UC-sets need not be a UC-set.

Recall that a set H of positive integers is called a *Hadamard set* if there is a constant $r > 1$ so that, when H is enumerated in increasing order as $\{h_j\}_{j=1}^{J}$, then $h_{j+1} \geqslant rh_j$ for all j. Also, if E and F are two sets of integers then $E - F$ denotes the set of all integers of the form $m - n$ where $m \in E$ and $n \in F$.

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THEOREM. *Let H be an infinite Hadamard set. Then $H - H$ is a UC-set, but it is not a CUC-set.*

PROOF. Let $E = H - H$. To show that E is a UC-set, it suffices, by [9, Theorem 2], to show that the positive and negative parts of E are both UC-sets. Since E is symmetric, it is enough to do this for the positive part of E. Finally, by [9, Theorem 3], it is enough to show that

$$\sup_{N>0} \kappa(E \cap [N, 2N]) < \infty.$$

To this end, enumerate H in increasing order as $\{h_j\}_{j=1}^{\infty}$, and let $r > 1$ be as in the definition of Hadamard set. Fix a positive integer N, and consider the indices j for which, for some index $i < j$, the difference $h_j - h_i$ lies in the interval $[N, 2N]$. Let J be the smallest such index j; then $h_J > N$. On the other hand, if j is any such index, then, in particular,

$$2N \geqslant h_j - h_{j-1} \geqslant (r-1)h_{j-1} \geqslant (r-1)r^{j-1-J}h_J > (r-1)r^{j-J-1}N.$$

Thus, $j - J - 1 < \log[2/(r-1)]/\log r = L(r)$, say. It follows that there are at most $L(r) + 1$ such indices j, and hence that $E \cap [N, 2N]$ is included in the union of at most $L(r) + 1$ translates of the set $-H$. Therefore there is a constant $C(r)$ so that $E \cap [N, 2N]$ has Sidon constant at most $C(r)$, and $\kappa(E \cap [N, 2N]) \leqslant C(r)$ also. Thus, E is indeed a UC-set.

To see that E is *not* a CUC-set, fix a positive integer M, and consider the Hilbert matrix $\{A_{m,n}\}_{m,n=1}^{M}$ given by letting

$$A_{m,n} = \begin{cases} 0 & \text{if } m = n, \\ \dfrac{1}{m-n} & \text{otherwise.} \end{cases}$$

Recall [3, Example 5.7] that the norm of A, as an operator on l^2, is at most π. Given a number θ in the interval $[0, 2\pi)$, let $v(\theta)$ be the vector in C^M with jth component $v_j(\theta) = \exp(ih_j\theta)$ for all j, and let

$$f(\theta) = (v(\theta), Av(\theta)) = \sum_{m,n=1, m\neq n}^{M} \frac{1}{m-n} \exp[i(h_m - h_n)\theta].$$

Then f is an $(H - H)$-polynomial. Moreover,

$$|f(\theta)| \leqslant \|A\|(\|v(\theta)\|_2)^2 \leqslant \pi M$$

for all θ, so that $\|f\|_\infty \leqslant \pi M$. On the other hand,

$$\begin{aligned} \sum_{k>0} \hat{f}(k) &= \sum_{1\leqslant n<m\leqslant M} \frac{1}{m-n} = \sum_{j=1}^{M-1} (M-j)\frac{1}{j} \\ &= (M-1) + M\left(\sum_{j=2}^{M-1} \frac{1}{j}\right) - \sum_{j=2}^{M-1} 1 \\ &> M(\log M - \log 2) \geqslant (1/\pi)\|f\|_\infty \log(M/2). \end{aligned}$$

Let $N = h_M$, and let $g(\theta) = f(\theta)\exp(-iN\theta)$. Then g is an $(E - h_M)$-polynomial, and

$$\|S_N(g)\|_\infty \geqslant \left|\sum_{|n|\leqslant N} \hat{g}(n)\right| = \sum_{k>0} \hat{f}(k) \geqslant (1/\pi)\|g\|_\infty \log(M/2).$$

Therefore, $\kappa(E - h_M) \geqslant (1/\pi)\log(M/2)$ for all M, and E is *not* a CUC-set. See Remark 3 below for another proof that E is not a CUC-set.

REMARK. 1. Now that we have examples of UC-sets that are not CUC-sets, we can, as pointed out in [**8**] easily construct pairs of UC-sets whose union is not a UC-set. Indeed, let $H = \{h_j\}_{j=1}^\infty$ be a Hadamard set for which in fact $h_{j+1} \geqslant 2h_j$ for all j; given H, let

$$A = \{m : m = h_i - h_j + h_k \text{ where } i > j > k\}.$$

Then, by the proof of Proposition 2 of [**8**], the sets A and B are both UC-sets, but $A \cup B$ is not a UC-set.

2. A related example is suggested by an observation on p. 283 of [**9**]. Suppose that, in the example above, the integers h_j are all even, and let $C = A \cup (B - 1)$. Then C is a UC-set, as is $C + 1$, but $C \cup (C + 1)$ is not a UC-set, because it includes $A \cup B$.

3. The second part of the proof of our theorem actually shows that if E and F are two infinite sets of positive integers, then $E - F$ is not a CUC-set. Here is an amusing alternate proof of this implication. If $E - F$ were a CUC-set, then, by [**8**, Proposition 1], there would exist a measure μ such that

$$\hat{\mu}(n) = \begin{cases} 1 & \text{if } n \in E - F \text{ and } n > 0, \\ 0 & \text{if } n \in E - F \text{ and } n < 0. \end{cases}$$

Enumerate the sets E and F as $\{m_j\}_{j=1}^\infty$ and $\{n_j\}_{j=1}^\infty$ respectively, and, for each index j, let ϕ_j and ψ_j be the functions on $[0, 2\pi)$ given respectively by $t \mapsto \exp(-im_j t)$ and $t \mapsto \exp(+in_j t)$; then

$$\int_T \phi_j(t)\psi_k(t)\, d\mu(t) = \hat{\mu}(m_j - n_k) = \begin{cases} 1 & \text{if } m_j > n_{ki}, \\ 0 & \text{if } m_j < n_k. \end{cases}$$

Let ϕ and ψ be accumulation points in $L^\infty(d|\mu|)$ of the respective sequences $\{\phi\}_{j=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$. Then $\int \phi\psi\, d\mu$ can be approximated arbitrarily well by integrals of the form $\int \phi\psi_k\, d\mu$, and any such integral can in turn be approximated arbitrarily well by integrals of the form $\int \phi_j\psi_k\, d\mu$, where $m_j > n_k$; hence $\int \phi\psi\, d\mu = 1$. On the other hand, by approximating ϕ first by ϕ_j, and then approximating ψ by ψ_k, where $n_k > m_j$, one sees that $\int \phi\psi\, d\mu$ must also be equal to 0. This contradiction shows that there is no such measure μ, and hence that $E - F$ is not a CUC-set.

4. It follows from the implication above that, if E, F, and G are infinite sets of positive integers, then $E - F + G$ and $E - F - G$ are not UC-sets. This contrasts with the fact [**9**, Theorem 7] that, if E is a *Paley set*, in other words a union of finitely-many Hadamard sets, then $E + E + E$ is a UC-set, as is $E + E + E + E$, etc. In view of our main theorem, one might ask if $E - E$ must be a UC-set whenever E is a Paley set; the answer is "no", because there are pairs (E_1, E_2) of Hadamard sets for which $E_1 - E_2$ consists of all integers [**5**, p. 69]. In a similar

vein, one can ask [**9**, p. 283] whether $E + E$ must be a UC-set whenever E is a dissociate set of positive integers; see [**5**, p. 19] for a definition of "dissociate". The answer is again "no"; the proof uses Hilbert matrices, and will appear in [**2**].

5. It is known [**9**, Lemma 6] that subsets of the positive integers that are UC-sets are also CUC-sets. Therefore, the sets A and B considered in Remark 1 provide an example of a pair of CUC-sets whose union is not even a UC-set.

6. Fix a strictly increasing sequence $N = \{N_j\}_{j=1}^{\infty}$ of positive integers, and call a set E a *UC(N)-set* if for every function f in C_E, the sequence $\{S_{N_j}(f)\}_{j=1}^{\infty}$ converges uniformly. It was pointed out by B.-Y. Ng [**6**] that when $N = \{2^j\}_{j=1}^{\infty}$ there are pairs of UC(N)-sets whose union is not a UC(N)-set. In fact, the methods of the present paper show that, for each such strictly increasing sequence N, there is a pair of CUC-sets A and B, as in Remark 1, whose union is not a UC(N)-set.

7. Given an index p in the interval $[1, \infty)$, and a set E of integers, let L_E^p be the subspace of all E-functions in $L^p(T)$. Call E an *L^pC-set* if $\|S_N(f) - f\|_p \to 0$ as $n \to \infty$ for all f in L_E^p. Again, E is an L^pC-set if and only if the quantity $\kappa_p(E) = \sup_{f,N}\{\|S_N(f)\|_p\colon f \in L_E^p,\ \|f\|_p = 1,\ N$ a nonnegative integer$\}$ is finite. Finally, call E a *CL^pC-set* if the sequence $\{\kappa_p(E + n)\}_{n=-\infty}^{\infty}$ is bounded. These notions are not interesting when $1 < p < \infty$, because the M. Riesz theorem shows that every set is a CL^pC-set in that case. It is not known, however, whether the classes of L^1C-sets or CL^1C-sets are closed under finite unions. S. Hartman [private communication] has observed that our examples shed some light on the relations between these classes and the classes of UC-sets and CUC-sets. First, it is easy to see that every UC-set is an L^1C-set, and that every CUC-set is a CL^1C-set. The examples given in Remark 1 show that the converses to these implications are false. Indeed, it is known [**1**, Theorem 5] that the set $A \cup B$ is a $\Lambda(2)$-set; it follows that $A \cup B$ is a CL^1C-set although it is not a UC-set.

8. I am pleased to acknowledge that the main theorem of this paper resulted from conversations with Ron Blei and Gordon Woodward. I have also benefited from helpful comments by S. Hartman, E. Sawyer, P. M. Soardi, and G. Travaglini.

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Department of Mathematics, University of British Columbia, Vancouver, Canada V6T 1Y4