TWO UC-SETS WHOSE UNION IS NOT A UC-SET

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Abstract. It is shown that the union of two sets of uniform convergence need not be a set of uniform convergence.

We use the standard terminology of harmonic analysis on the unit circle as in [4]. We recall some notions discussed in [8] and [9], and in the references cited in these papers.

Definition. Given a subset $E$ of the integers, call an integrable function $f$, on the circle, an $E$-function if $\hat{f}(n) = 0$ for all integers $n$ outside $E$, and denote the space of continuous $E$-functions by $C_E$. Call $E$ a set of uniform convergence, or a UC-set, if every function in $C_E$ has a uniformly convergent Fourier series.

The union problem for UC-sets is mentioned as an open problem in [5, p. 86; 9, p. 283]. To solve it, we need a few more facts about UC-sets. It is known that $E$ is a UC-set if and only if there is a constant $\kappa$ so that, for each function $f$ in $C_E$, the partial sums $S_N(f)$ of the Fourier series of $f$ satisfy the inequality $\|S_N(f)\|_\infty \leq \kappa \|f\|_\infty$ for all nonnegative integers $N$. Furthermore, when $E$ is a UC-set, there is a smallest value of $\kappa$ for which the inequality above holds for all such $f$ and $N$; this smallest value of $\kappa$ is called the UC-constant of $E$, and is denoted by $\kappa(E)$. If $E$ is a UC-set, then so is every translate of $E$, but it turns out that the translates of a UC-set do not all have the same UC-constant.

Definition. Call $E$ a CUC-set, or a set of completely uniform convergence if $E$ is a UC-set with the additional property that the sequence $(\kappa(E + n))_{n=-\infty}^\infty$ is bounded.

This notion was introduced, independently by G. Travaglini [9, Lemma 6] and F. Ricci [7, p. 426]. In [8], P. M. Soardi and Travaglini gave some nontrivial examples of CUC-sets, and they showed that if there is a UC-set that is not a CUC-set, then there is a pair of UC-sets whose union is not a UC-set. In the present paper, we exhibit a class of UC-sets that are not CUC-sets, thereby showing that the union of two UC-sets need not be a UC-set.

Recall that a set $H$ of positive integers is called a Hadamard set if there is a constant $r > 1$ so that, when $H$ is enumerated in increasing order as $\{h_j\}_{j=1}^\infty$, then $h_{j+1} \geq rh_j$ for all $j$. Also, if $E$ and $F$ are two sets of integers then $E - F$ denotes the set of all integers of the form $m - n$ where $m \in E$ and $n \in F$. 

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Theorem. Let \( H \) be an infinite Hadamard set. Then \( H - H \) is a UC-set, but it is not a CUC-set.

Proof. Let \( E = H - H \). To show that \( E \) is a UC-set, it suffices, by [9, Theorem 2], to show that the positive and negative parts of \( E \) are both UC-sets. Since \( E \) is symmetric, it is enough to do this for the positive part of \( E \). Finally, by [9, Theorem 3], it is enough to show that

\[
\sup_{N > 0} \kappa(E \cap [N, 2N]) < \infty.
\]

To this end, enumerate \( H \) in increasing order as \( \{h_j\}_{j=1}^\infty \), and let \( r > 1 \) be as in the definition of Hadamard set. Fix a positive integer \( N \), and consider the indices \( j \) for which, for some index \( i < j \), the difference \( h_j - h_i \) lies in the interval \([N, 2N]\). Let \( J \) be the smallest such index \( j \); then \( h_j > N \). On the other hand, if \( j \) is any such index, then, in particular,

\[
2N > h_j - h_{j-1} > (r - 1)h_{j-1} > (r - 1)r^{j-1} - h_j > (r - 1)r^{j-1}N.
\]

Thus, \( j - J - 1 < \log(2/(r - 1))/\log r = L(r) \), say. It follows that there are at most \( L(r) + 1 \) such indices \( j \), and hence that \( E \cap [N, 2N] \) is included in the union of at most \( L(r) + 1 \) translates of the set \(-H\). Therefore there is a constant \( C(r) \) so that \( E \cap [N, 2N] \) has Sidon constant at most \( C(r) \), and \( \kappa(E \cap [N, 2N]) < C(r) \) also. Thus, \( E \) is indeed a UC-set.

To see that \( E \) is not a CUC-set, fix a positive integer \( M \), and consider the Hilbert matrix \( \{A_{m,n}\}_{m,n=1}^M \) given by letting

\[
A_{m,n} = \begin{cases} 
0 & \text{if } m = n, \\
\frac{1}{m - n} & \text{otherwise}.
\end{cases}
\]

Recall [3, Example 5.7] that the norm of \( A \), as an operator on \( l^2 \), is at most \( \pi \). Given a number \( \theta \) in the interval \([0, 2\pi)\), let \( v(\theta) \) be the vector in \( CM \) with \( j \)th component \( v_j(\theta) = \exp(ih_j\theta) \) for all \( j \), and let

\[
f(\theta) = (v(\theta), Av(\theta)) = \sum_{m,n=1, m\neq n}^M \frac{1}{m - n} \exp[i(h_m - h_n)\theta].
\]

Then \( f \) is an \((H - H)\)-polynomial. Moreover,

\[
|f(\theta)| < \|A\| \|v(\theta)\|_2^2 < \pi M
\]

for all \( \theta \), so that \( \|f\|_\infty < \pi M \). On the other hand,

\[
\sum_{k > 0} \hat{f}(k) = \sum_{1 < n < m < M} \frac{1}{m - n} = \sum_{j=1}^{M-1} (M - j) \frac{1}{j}.
\]

\[
= (M - 1) + M \left( \sum_{j=2}^{M-1} \frac{1}{j} \right) - \sum_{j=2}^{M-1} 1
\]

\[
> M(\log M - \log 2) > (1/\pi)\|f\|_\infty \log(M/2).
\]
Let $N = h_M$, and let $g(\theta) = f(\theta)\exp(-iN\theta)$. Then $g$ is an $(E - h_M)$-polynomial, and

$$\|S_N(g)\|_\infty \geq \left| \sum_{|n| < N} \hat{g}(n) \right| = \sum_{k > 0} \hat{f}(k) > \left(1/\pi\right)\|g\|_\infty \log(M/2).$$

Therefore, $\kappa(E - h_M) > (1/\pi)\log(M/2)$ for all $M$, and $E$ is not a CUC-set. See Remark 3 below for another proof that $E$ is not a CUC-set.

**Remark.** 1. Now that we have examples of UC-sets that are not CUC-sets, we can, as pointed out in [8] easily construct pairs of UC-sets whose union is not a UC-set. Indeed, let $H = \{h_j\}_{j=1}^\infty$ be a Hadamard set for which in fact $h_{j+1} > 2h_j$ for all $j$; given $H$, let

$$A = \{ m: m = h_i - h_j + h_k \text{ where } i > j > k \}.$$ 

Then, by the proof of Proposition 2 of [8], the sets $A$ and $B$ are both UC-sets, but $A \cup B$ is not a UC-set.

2. A related example is suggested by an observation on p. 283 of [9]. Suppose that, in the example above, the integers $h_j$ are all even, and let $C = A \cup (B - 1)$. Then $C$ is a UC-set, as is $C + 1$, but $C \cup (C + 1)$ is not a UC-set, because it includes $A \cup B$.

3. The second part of the proof of our theorem actually shows that if $E$ and $F$ are two infinite sets of positive integers, then $E - F$ is not a CUC-set. Here is an amusing alternate proof of this implication. If $E - F$ were a CUC-set, then, by [8, Proposition 1], there would exist a measure $\mu$ such that

$$\mu(n) = \begin{cases} 
1 & \text{if } n \in E - F \text{ and } n > 0, \\
0 & \text{if } n \in E - F \text{ and } n < 0.
\end{cases}$$

Enumerate the sets $E$ and $F$ as $\{m_j\}_{j=1}^\infty$ and $\{\eta_j\}_{j=1}^\infty$ respectively, and, for each index $j$, let $\phi_j$ and $\psi_j$ be the functions on $[0, 2\pi)$ given respectively by $t \mapsto \exp(-im_j t)$ and $t \mapsto \exp(+im_j t)$; then

$$\int_T \phi_j(t)\psi_k(t) \, d\mu(t) = \mu(m_j - n_k) = \begin{cases} 
1 & \text{if } m_j > n_k, \\
0 & \text{if } m_j < n_k.
\end{cases}$$

Let $\phi$ and $\psi$ be accumulation points in $L^\infty(d|\mu|)$ of the respective sequences $\{\phi_j\}_{j=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$. Then $\phi\psi \, d\mu$ can be approximated arbitrarily well by integrals of the form $\phi\psi_k \, d\mu$, and any such integral can in turn be approximated arbitrarily well by integrals of the form $\phi_k \psi_k \, d\mu$, where $m_j > n_k$; hence $\phi\psi \, d\mu = 1$. On the other hand, by approximating $\phi$ first by $\phi_j$, and then approximating $\psi$ by $\psi_k$, where $n_k > m_j$, one sees that $\phi\psi \, d\mu$ must also be equal to 0. This contradiction shows that there is no such measure $\mu$, and hence that $E - F$ is not a CUC-set.

4. It follows from the implication above that, if $E$, $F$, and $G$ are infinite sets of positive integers, then $E - F + G$ and $E - F - G$ are not UC-sets. This contrasts with the fact [9, Theorem 7] that, if $E$ is a Paley set, in other words a union of finitely-many Hadamard sets, then $E + E + E$ is a UC-set, as is $E + E + E + E$, etc. In view of our main theorem, one might ask if $E - E$ must be a UC-set whenever $E$ is a Paley set; the answer is “no”, because there are pairs $(E_1, E_2)$ of Hadamard sets for which $E_1 - E_2$ consists of all integers [5, p. 69]. In a similar
vein, one can ask [9, p. 283] whether $E + E$ must be a UC-set whenever $E$ is a dissociate set of positive integers; see [5, p. 19] for a definition of “dissociate”. The answer is again “no”; the proof uses Hilbert matrices, and will appear in [2].

5. It is known [9, Lemma 6] that subsets of the positive integers that are UC-sets are also CUC-sets. Therefore, the sets $A$ and $B$ considered in Remark 1 provide an example of a pair of CUC-sets whose union is not even a UC-set.

6. Fix a strictly increasing sequence $N = \{N_j\}_{j=1}^\infty$ of positive integers, and call a set $E$ a $UC(N)$-set if for every function $f$ in $C_E$, the sequence $\{S_N(f)\}_{j=1}^\infty$ converges uniformly. It was pointed out by B.-Y. Ng [6] that when $N = \{2^j\}_{j=1}^\infty$ there are pairs of $UC(N)$-sets whose union is not a $UC(N)$-set. In fact, the methods of the present paper show that, for each such strictly increasing sequence $N$, there is a pair of CUC-sets $A$ and $B$, as in Remark 1, whose union is not a $UC(N)$-set.

7. Given an index $p$ in the interval $[1, \infty)$, and a set $E$ of integers, let $L_p^E$ be the subspace of all $E$-functions in $L_p^1(T)$. Call $E$ an $L_p^C$-set if $\|S_n(f) - f\|_p \to 0$ as $n \to \infty$ for all $f$ in $L_p^E$. Again, $E$ is an $L_p^C$-set if and only if the quantity $K_p(E) = \sup_{f, n} \{\|S_n(f)\|_p : f \in L_p^E, \|f\|_p = 1, N$ a nonnegative integer} is finite. Finally, call $E$ a $CL_p^C$-set if the sequence $\{K_p(E + n)\}_{n=1}^\infty$ is bounded. These notions are not interesting when $1 < p < \infty$, because the M. Riesz theorem shows that every set is a $CL_p^C$-set in that case. It is not known, however, whether the classes of $L_p^C$-sets or $CL_p^C$-sets are closed under finite unions. S. Hartman [private communication] has observed that our examples shed some light on the relations between these classes and the classes of UC-sets and CUC-sets. First, it is easy to see that every UC-set is an $L_1^C$-set, and that every CUC-set is a $CL_1^C$-set. The examples given in Remark 1 show that the converses to these implications are false. Indeed, it is known [1, Theorem 5] that the set $A \cup B$ is a $\Lambda(2)$-set; it follows that $A \cup B$ is a $CL_1^C$-set although it is not a UC-set.

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