ON A THEOREM OF BAKER, LAWRENCE AND ZORZITTO

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Abstract. The result of J. Baker, J. Lawrence and F. Zorzitto on the stability of the equation \( f(x + y) = f(x)f(y) \) is generalized by proving the following theorem: if \( G \) is a semigroup and \( V \) is a right invariant linear space of complex valued functions on \( G \), and if \( f, m \) are complex valued functions on \( G \) for which the function \( x \rightarrow f(xy) - f(x)m(y) \) belongs to \( V \) for every \( y \) in \( G \), then either \( f \) is in \( V \) or \( m \) is exponential.

In [1] J. Baker, J. Lawrence and F. Zorzitto, solving a problem of E. Lukacs on the stability of the functional equation \( f(x + y) = f(x)f(y) \) proved that if \( f \) is a function from a vector space to the real numbers satisfying

\[
|f(x + y) - f(x)f(y)| < \delta
\]

then \( f \) is either bounded or exponential. This result was also generalized and simplified in [2]. Here we generalize this result in another way.

Let \( G \) be a semigroup and \( V \) be a linear space of complex valued functions on \( G \). Then \( V \) is called right invariant if \( f \) belongs to \( V \) implies that the function \( x \rightarrow f(xy) \) belongs to \( V \) for every \( y \) in \( G \). Similarly, we can define left invariant linear spaces, and we call \( V \) invariant if it is right and left invariant.

The complex valued function \( m : G \rightarrow \mathbb{C} \) (\( \mathbb{C} \) denotes the set of complex numbers) is called an exponential if for every \( x, y \) in \( G \) we have

\[
m(xy) = m(x)m(y).
\]

Our main result is the following

**Theorem.** Let \( G \) be a semigroup and \( V \) be a right invariant linear space of complex valued functions on \( G \). Let \( f, m : G \rightarrow \mathbb{C} \) be complex valued functions for which the function \( x \rightarrow f(xy) - f(x)m(y) \) belongs to \( V \) for every \( y \) in \( G \). Then either \( f \) is in \( V \) or \( m \) is an exponential.

**Proof.** Suppose that \( m \) is not an exponential. Then there exist \( y, z \) in \( G \) with the property \( m(yz) - m(y)m(z) \neq 0 \). On the other hand we have, for all \( x \) in \( G \),

\[
f(xyz) - f(xy)m(z) = [f(xyz) - f(x)m(yz)] - m(z)[f(xy) - f(x)m(y)] + f(x)[m(yz) - m(y)m(z)]
\]

and hence

\[
f(x) = [(f(xyz) - f(xy)m(z)) - (f(xyz) - f(x)m(yz))] + m(z)(f(xy) - m(y)f(x)) \cdot [m(yz) - m(y)m(z)]^{-1}.
\]
Here the right-hand side as a function of $x$ belongs to $V$, and hence so does $f$.

**Corollary.** Let $G$ be a semigroup with identity and $V$ be an invariant linear space of complex valued functions on $G$. Let $f, m : G \rightarrow \mathbb{C}$ be complex valued functions for which the functions $x \rightarrow f(xy) - f(x)m(y)$ and $y \rightarrow f(xy) - f(x)m(y)$ belong to $V$ for every $y$ in $G$ and $x$ in $G$, respectively. Then either $f$ is in $V$ or $m$ is an exponential and $f = f(1)m$.

**Proof.** Suppose that $f$ is not in $V$. Then by the preceding theorem, $m$ is an exponential. On the other hand, the function $y \rightarrow f(y) - f(1)m(y)$ is in $V$ and for $x, y$ in $G$ we have

$$f(xy) - f(1)m(xy) = f(xy) - f(x)m(y) + f(x)m(y) - f(1)m(x)m(y)$$

$$= f(xy) - f(x)m(y) + [f(x) - f(1)m(x)]m(y).$$

If there is an $x_0$ in $G$ for which $f(x_0) \neq f(1)m(x_0)$, then $m$ belongs to $V$ and so does $f$, which is a contradiction. Hence $f = f(1)m$ which was to be proved.

**Remark.** Here we make clear how the corollary generalizes the cited result. Let $G$ be an Abelian semigroup with identity and $V$ be the space of bounded complex valued functions on $G$. If $f, m : G \rightarrow \mathbb{C}$ are functions for which there exist $M_1, M_2 : G \rightarrow [0, +\infty)$ such that

$$|f(xy) - f(x)m(y)| < \min(M_1(x), M_2(y))$$

for all $x, y$ in $G$ then either $f$ is bounded or $m$ is exponential and $f = f(1)m$.

**Example.** Let $G$ be a commutative topological group and $f : G \rightarrow \mathbb{C}$ be such that $x \rightarrow f(x + y) - f(x)f(y)$ is continuous for each $y$ in $G$. Then either $f$ is continuous or exponential. Other interesting examples can be constructed by taking $V$ to be the class of measurable or integrable functions on appropriate groups.

**References**


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