DERIVATIVES OF $H^p$ FUNCTIONS

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ABSTRACT. We prove that if $\{z^n\}$ is uniformly separated and $f \in H^p$, then 
\[ \left\{ f^{(k)}(z^n)(1 - |z^n|^2)^{k+1/p} \right\}_{n=1}^\infty \in l^p \] 
for $k = 1, 2, \ldots$.

We give a simple proof of

LEMMA. Let $\{z^n\}$ be uniformly separated and $f \in H^p$. For $k = 1, 2, \ldots$ we have 
\[ \left\{ f^{(k)}(z^n)(1 - |z^n|^2)^{k+1/p} \right\}_{n=1}^\infty \in l^p. \]

$H^p$ is the Hardy space of the unit disc $D$. A sequence $\{z^n\} \in D$ is called 
uniformly separated if 
\[ \inf_n \prod_{m \neq n} \left| \frac{z_n - z_m}{1 - \bar{z}_n z_m} \right| > 0. \]

A technical proof of the lemma was given in [2]. There it was also proved that 
every $l^p$ sequence is obtained in this way. When [2] was published, the result was 
already known in the Soviet Union (see, for instance, F. A. Shamoian’s paper [3]). 
Inspired by this paper we prove the lemma.

For small $\tau$ let $D_n = \{z: |z - z_n| \leq \tau(1 - |z_n|)\}$. A simple computation using the 
pseudohyperbolic metric $\eta(a, b) = \frac{|a - b|}{1 - \bar{a}b}$ proves that $z^*_n \in D_n \Rightarrow \{z^*_n\}$ 
is uniformly separated. By Cauchy’s formula

\[ |f^{(k)}(z^n)| = \frac{k!}{2\pi i} \int_{\partial D_n} \frac{f(\xi)}{(\xi - z_n)^{k+1}} d\xi \leq A(1 - |z_n|^2)^{-k} \max_{\xi \in D_n} |f(\xi)| \]

\[ = A(1 - |z_n|^2)^{-k} |f(z^*_n)|. \]

Hence

\[ |f^{(k)}(z^n)(1 - |z_n|^2)^{k+1/p}| \leq A |f(z^*_n)|(1 - |z_n|^2)^{1/p} \]

\[ \leq A \cdot B |f(z^*_n)|(1 - |z^*_n|^2)^{1/p} \]

where $B$ is seen to be independent of $n$. Since $\{z^*_n\}$ is uniformly separated, 
the lemma follows from the well-known interpolation theorem of Shapiro and Shields [1].

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REFERENCES


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