PROOF OF A CONJECTURE OF ERDÖS
ABOUT THE LONGEST POLYNOMIAL

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Abstract. In 1939 P. Erdös conjectured that the Chebyshev polynomial \( T_n(x) \) has a maximal arc-length in \([-1, 1]\) among the polynomials of degree \( n \) which are bounded by 1 in \([-1, 1]\). We prove this conjecture for every natural \( n \).

1. Introduction. P. Erdös proved in [2] that the function \( \cos nt \) has a maximal arc-length in \([-\pi, \pi]\) among all trigonometric polynomials of order \( n \) with a uniform norm equal to 1. He has conjectured that the Chebyshev polynomial

\[
T_n(x) = \cos(n \text{ arc } \sin x), \quad -1 < x < 1,
\]

is the unique extremal function in the corresponding analogous problem in the set \( \pi_n \) of algebraic polynomials of degree less than or equal to \( n \).

Denote by \( l(f) \) the arc-length of the function \( f \) in \([-1, 1] \), i.e.,

\[
l(f) := \int_{-1}^{1} \left[ 1 + f'^2(x)^2 \right]^{1/2} \, dx.
\]

Set \( \|f\| = \max\{|f(x)|; -1 < x < 1\} \).

Conjecture of Erdös. The quantity

\[
\sup\{l(f); f \in \pi_n, \|f\| < 1\} \quad (n = 1, 2, \ldots)
\]

is attained if and only if \( f = \pm T_n \).

This conjecture has remained an open problem for over 40 years. In a recent work Szabados [4] showed that the polynomials \( T_n \) are asymptotically extremal as \( n \to \infty \). We prove here the conjecture of Erdös for each natural number \( n \). Our proof is based on a variational approach.

2. Explanatory statement. The problem of Erdös is set for the domain \([-1, 1] \times [-1, 1] \), i.e., for the class of polynomials \( f \in \pi_n \) such that \( |f(x)| < 1 \) if \( |x| < 1 \). One may guess that the solution \( f(x) \) in this particular case suffices to construct the solution \( f(M; x) \) of the corresponding problem about the longest polynomial in the domain \([-1, 1] \times [-M, M] \) for every \( M > 0 \). One even suggests the following simple formula:

\[
(*) \quad f(M; x) = Mf(x).
\]

It turns out (see Theorem 1) that \((*)\) is actually true. But this is not evident. The problem \((*)\) is as difficult as that of Erdös. In any case, the relation \((*)\) yields easily the conjecture of Erdös. Indeed, suppose that \((*)\) holds for every \( M > 0 \).
Then
\[ \frac{1}{M} \int_{-1}^{1} \left[ 1 + M^2 g^2(x) \right]^{1/2} dx \leq \frac{1}{M} \int_{-1}^{1} \left[ 1 + M^2 f^2(x) \right]^{1/2} dx \]
for each \( M > 0 \), provided \( g \in \pi_n \), and \( \|g\| \leq 1 \). If we let \( M \) tend to infinity, we get
\[ \int_{-1}^{1} |g'(x)| \, dx \leq \int_{-1}^{1} |f'(x)| \, dx. \]
Thus, \( f \) should have a maximal variation in \([-1, 1]\). Therefore \( f = \pm T_n \).

Finally, note that the problem on an arbitrary interval \([a, b]\) is easily reduced to the problem on \([-1, 1]\) by a linear transformation.

3. Main result. In what is to follow, let \( M \) be a fixed positive number. With every natural number \( n \) we associate the set \( \Omega_n \subset \pi_n \) which is defined as follows. The polynomial \( f \in \pi_n \) belongs to \( \Omega_n \) if there exist \( m + 1 \) points \( \{x_i\}_{i=0}^m \) \((m \in \{1, \ldots, n\})\) such that

\[ -1 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1, \]
\[ |f(x_i)| = M, \quad i = 0, \ldots, m, \]
\[ f(x_i) = -f(x_{i+1}), \quad i = 0, \ldots, m - 1 \]
and \( f(x) \) is a monotone function in \([x_i, x_{i+1}]\), \( i = 0, \ldots, m - 1 \). It is clear that \( \|f\| = M \) if \( f \in \Omega_n \).

The basic idea of our proof is presented in the following lemma.

**Lemma 1.** Suppose that \( f \in \pi_n \), \( \|f\| = M \) and
\[ l(f) = \sup \{ l(g) : g \in \pi_n, \|g\| \leq M \}. \]

Then \( f \in \Omega_n \).

**Proof.** Without loss of generality we assume that \( f(x) > 0 \) for each sufficiently large \( x > 0 \). Denote by \( \{x_i\}_{i=0}^{m-1} \) the distinct zeros of \( f'(x) \) in \((-1, 1)\). Obviously \( m < n \). Set, for convenience, \( x_0 = -1, x_m = 1, \omega(x) = f'(x) \). We shall show that
\[ f(x_i) = (-1)^{m-1} M, \quad i = 0, \ldots, m. \]
This implies that \( f \in \Omega_n \).

Introduce the polynomials
\[ g_i(x) = (x^2 - 1)\omega(x) / (x - x_i), \quad i = 0, \ldots, m. \]
We intend to estimate the arc-length \( \sigma_i(\varepsilon) := l(f + \varepsilon g_i) \) for small \( \varepsilon \). Our first task is to show that
\[ \sigma_i'(0) > 0 \]
for \( i = 0, \ldots, m \). It is seen that
\[ \sigma_i'(0) = \int_{-1}^{1} \frac{\omega(x)}{\left[ 1 + \omega^2(x) \right]^{1/2}} g_i'(x) \, dx. \]
In the case \( i = 0 \) a straightforward calculation gives
\[ \sigma_0'(0) = 2 \left[ 1 + \omega^2(-1) \right]^{1/2} - \int_{-1}^{1} \left[ 1 + \omega^2(x) \right]^{-1/2} \, dx > 0. \]
Similarly, \( \sigma''(0) > 0 \). Now suppose that \( 1 < i < m - 1 \). Integrating by parts, we get

\[
\sigma'(0) = \int_{-1}^{1} \frac{x^2 - 1}{x - x_i} \left[ 1 + \omega^2(x) \right]^{-1/2} \, dx.
\]

The integrand is a continuous function in \([-1, 1]\). Therefore \( \sigma'(0) < \infty \) and \( \sigma'(0) = \lim\{ \mathcal{T}_i(\delta): \delta \to 0 \} \) where

\[
\mathcal{T}_i(\delta) = \int_{\Omega_i(\delta)} \frac{x^2 - 1}{x - x_i} \left[ 1 + \omega^2(x) \right]^{-1/2} \, dx
\]

and \( \Omega(\delta) := [-1, x_i - \delta] \cup [x_i + \delta, 1] \). Next we calculate \( \mathcal{T}_i(\delta) \). Observe first that \( \omega(x_i \pm \delta) = O(\delta) \). This yields, for instance, by Taylor’s formula, that

\[
\left( 1 + \omega^2(x) \right)^{-1/2} = 1 + O(\delta^2).
\]

Further, by the mean-value theorem for integrals, there exist points \( \xi_i = \xi_i(\delta) \in [-1, x_i - \delta] \) and \( \xi_{i+1} = \xi_{i+1}(\delta) \in [x_i, x_{i+1}, 1] \) such that

\[
\begin{align*}
\int_{-1}^{-1-x_i-\delta} \left[ 1 + \omega^2(x) \right]^{-1/2} (x - x_i)^2 \, dx & = c_1(\delta) \left[ 1/\delta - 1/(1 + x_i) \right], \\
\int_{x_i}^{1} \left[ 1 + \omega^2(x) \right]^{-1/2} (x - x_i)^2 \, dx & = c_2(\delta) \left[ 1/\delta - 1/(1 - x_i) \right]
\end{align*}
\]

where \( c_j(\delta) = \left[ 1 + \omega^2(\xi_j) \right]^{-1/2}, j = 1, 2 \). Obviously

\[
0 < c_j(\delta) < 1, \quad j = 1, 2.
\]

Let us set, for convenience, \( A(\delta) = \int_{\Omega(\delta)} \left[ 1 + \omega^2(x) \right]^{-1/2} \, dx \). Now, taking into account the relations (3) and (4), after integration by parts, we obtain

\[
\mathcal{T}_i(\delta) = \left[ (x^2 - 1)/(x - x_i) \right] \left[ 1 + \omega^2(x) \right]^{-1/2} \int_{x_i-\delta}^{x_i+\delta} \int_{\Omega(\delta)} \left[ 1 + \omega^2(x) \right]^{-1/2} \left( 1 + (1 - x_i^2)/(x - x_i)^2 \right) \, dx
\]

\[
= \delta^{-1} \left[ c_1(\delta) + c_2(\delta) - 2 \right] (x_i^2 - 1) + O(\delta) - A(\delta) - c_1(\delta)(x_i - 1) + c_2(\delta)(x_i + 1).
\]

But, as we have already mentioned, \( \mathcal{T}_i(\delta) \) has a limit as \( \delta \to 0 \). Then \( c_1(\delta) + c_2(\delta) \) must tend to 2, which combined with (5) implies \( c_j(\delta) \to 1, j = 1, 2, \) as \( \delta \to 0 \). Moreover, \( c_j(\delta) = 1 - \alpha_j(\delta) + o(\delta), j = 1, 2, \) with some constants \( \alpha_j > 0 \). Therefore

\[
\sigma'(0) = \lim\{ \mathcal{T}_i(\delta): \delta \to 0 \} = -A(0) + 2 - (\alpha_1 + \alpha_2)(x_i^2 - 1) > 0.
\]

Our claim (2) is proved.

Now, let us assume that \( f \) does not belong to \( \Omega_n \). Then there exists at least one point \( x_i \in \{ x_0, \ldots, x_m \} \) such that \( |f(x_i)| < M \). Consider the polynomial \( \varphi_{\epsilon}(x) := f(x) + \epsilon g(x) \). Evidently, \( l(\varphi_{\epsilon}) = \sigma_0(\epsilon) + \epsilon \sigma_0'(t_\epsilon) = l(f) + \epsilon \sigma_0'(t_\epsilon) \) where \( 0 < t_\epsilon < \epsilon \). But, according to (2), there exists an \( \epsilon_0 > 0 \) such that \( \sigma := \min\{ \sigma(t): 0 < t < \epsilon_0 \} > 0 \). Therefore

\[
l(\varphi_{\epsilon}) > l(f) + \epsilon \sigma
\]

for each \( \epsilon \in [0, \epsilon_0] \).
Let us estimate the uniform norm of $q_\varepsilon$ in $[-1, 1]$ for small $\varepsilon$. In order to do this, it suffices to investigate the function $q_\varepsilon(x)$ near the points $\{x_j\}$ for which $|f(x_j)| = M$. Let $x_k$ be such a point. Without loss of generality we may assume that $f(x_k) = M$. Suppose that $\varepsilon$ is chosen to satisfy the requirement $x_k \in [x_k - \varepsilon, x_k + \varepsilon] \cap [-1, 1] =: B(x_k; \varepsilon)$ for every $j \neq k$. Let $q_\varepsilon(x)$ attain its maximal value in the neighbourhood $B(x_k; \varepsilon)$ of $x_k$ at the point $z_k(\varepsilon)$. On expanding $q_\varepsilon(x)$ in a partial Taylor series around $x = x_k$, we get

$$q_\varepsilon(z_k(\varepsilon)) < M + \varepsilon \| g \| |z_k(\varepsilon) - x_k|$$

for sufficiently small $\varepsilon > 0$. It is not difficult to see that $|z_k(\varepsilon) - x_k| \to 0$ as $\varepsilon \to 0$. Then, in view of the last inequality, $\|q_\varepsilon\| < M + \varepsilon \delta(\varepsilon)$, where $\delta(\varepsilon)$ is a function which tends to zero as $\varepsilon \to 0$. Now construct the polynomial

$$\psi_\varepsilon(x) = \left(1 - \frac{\varepsilon \delta(\varepsilon)}{M + \varepsilon \delta(\varepsilon)}\right) q_\varepsilon(x).$$

Clearly, $\psi_\varepsilon \in \pi_n$ and $\|\psi_\varepsilon\| < M$. We shall show that $l(\psi_\varepsilon) > l(f)$ for small $\varepsilon > 0$. Indeed, since $L := \delta(L)/\delta|\lambda| > 0$, we have $l(\psi_\varepsilon) > l(q_\varepsilon) - (2L/M)\varepsilon \delta(\varepsilon)$ for small $\varepsilon > 0$. Next we apply (6) and get

$$l(\psi_\varepsilon) > l(f) + \left[\sigma - (2L/M) \delta(\varepsilon)\right] \varepsilon > l(f)$$

for sufficiently small $\varepsilon > 0$. Thus, $f$ is not extremal, a contradiction. Therefore $|f(x_i)| = M$ for $i = 0, \ldots, m$. Since $\{x_i\}_{i=0}^{m-1}$ are all distinct zeros of $f'(x)$ in $(-1, 1)$, we conclude that (1) is valid. The lemma is proved.

It remains to show that the extremal polynomial $f$ must have $n + 1$ points of alternation. For this, we give below an interesting property of the Chebyshev polynomial $T_n(x)$.

Let $\{\theta_k\}_{k=0}^n$ be the extremal points of $T_n(x)$ in $[-1, 1]$. It is well known (see Rivlin [3]) that $\theta_0 = -1, \theta_n = 1$ and $T_n(\theta_k) = (-1)^{n-k}, k = 0, \ldots, n$. Suppose that $f \in \Omega_n$ and $f'(x)$ has $m - 1$ distinct zeros $x_1, \ldots, x_{m-1}$ in $(-1, 1)$. Evidently, there is an $i \in \{0, \ldots, m - 1\}$ such that $x_i < 0 < x_{i+1}$. Consider the partition of $[-1, 1]$ into subintervals $[x_0, x_1], \ldots, [x_i, 0], [0, x_{i+1}], \ldots, [x_{m-1}, x_m]$ which we denote, for simplicity, by $I_0, \ldots, I_m$, respectively. Define the points $t_1$ and $t_2$ by the conditions

$$t_1 \in [\theta_i, \theta_{i+1}], \quad MT_n(t_1) = f(0),$$

$$t_2 \in [\theta_{i+n-m}, \theta_{i+n-m+1}], \quad MT_n(t_2) = f(0).$$

Denote the intervals $[\theta_0, \theta_1], \ldots, [\theta_i, t_1], [t_2, \theta_{i+n-m+1}], \ldots, [\theta_{n-1}, \theta_n]$ by $I_0^*, \ldots, I_m^*$. We shall refer to $I_k^*$ as the corresponding interval to $I_k$.

**Lemma 2.** Suppose that $f$ is a polynomial from the set $\Omega_n$ with $m + 1$ extremal points, $\alpha \in (-M, M)$ and $k \in \{0, \ldots, m\}$. Let the points $\xi$ and $\eta$ satisfy the conditions

$$\xi \in I_k^*, \quad MT_n(\xi) = \alpha, \quad \eta \in I_k, \quad f(\eta) = \alpha.$$

Then $|f'(\eta)| < M |T_n'(\xi)|$.

The assertion follows easily from a known extremal property of $\cos nt$. The proof is given with details in [1].
We are now prepared to prove the main theorem.

**Theorem 1.** Let \( n \) be an arbitrary natural number. Then, for each \( M > 0 \), the quantity

\[
\sup\{l(f): f \in \pi_n, \|f\| < M\}
\]

is attained if and only if \( f = \pm MT_n \).

**Proof.** Note first that the inequality \(|d| < |c|\) implies

\[
(1 + c^2)^{1/2} < (1 + d^2)^{1/2} + |c| - |d|.
\]

We shall make use of this in the sequel. Suppose that \( f \in \Omega_n \) and \([-1, 1] = I_0 \cup \cdots \cup I_m \) is the partition of \([-1, 1]\) induced by \( f \). Let the intervals \( I = [z_1, z_2] \) and \( I^* = [z_1^*, z_2^*] \) be corresponding. Denote by \( u(y) \) and \( v(y) \) the inverse functions of \( f(x) \) and \( MT_n(x) \) in \( I \) and \( I^* \), respectively. According to Lemma 2, we have \( |v'(y)| < |u'(y)| \) for each \( y \in (-M, M) \). Then, applying (7), we get

\[
\int_{-M}^{M} [1 + u'^2(y)]^{1/2} dy < \int_{-M}^{M} [1 + v'^2(y)]^{1/2} dy + \int_{-M}^{M} |u'(y)| dy - \int_{-M}^{M} |v'(y)| dy.
\]

Denote by \( l(g; K) \) the arc-length of \( g \) over the set \( K \). Then the above inequality means that \( l(f; I) < l(MT_n; I^*) + |z_2 - z_1| - |z_2^* - z_1^*| \). Summing for \( I = I_0, \ldots, I_m \), we obtain

\[
l(f) < l(MT_n; [-1, t_1] \cup [t_2, 1]) + t_2 - t_1 \leq l(MT_n).
\]

The equality holds if and only if \( t_1 = t_2 \), i.e., iff \( f = \pm MT_n \). The theorem is proved.

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**References**


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