

CARDINAL ARITHMETIC AND \aleph_1 -BOREL SETS

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ABSTRACT. It is shown to be consistent with $2^{\aleph_0} > \aleph_1$ that the smallest \aleph_2 -complete Boolean subalgebra of $\mathcal{P}(\mathbb{R})$ containing all closed sets is $\mathcal{P}(\mathbb{R})$. Some related results are also proved.

Introduction. The object of study of this paper is the set of \aleph_1 -Borel sets. The reader has no doubt already guessed the definition of an \aleph_1 -Borel set but if not he should consult §1. One objection which might be raised to such a study is that the concept of an \aleph_1 -Borel set is not nearly as natural as the concept of a Borel set. However certain classical results of descriptive set theory show that this is not entirely true. Sierpinski has shown that every Σ_2^1 -set is \aleph_1 -Borel and in fact is the union of \aleph_1 -many Borel sets [7]. Hausdorff has also established a relationship between Borel sets and the cardinal \aleph_1 by showing that the real line can be partitioned into \aleph_1 -many Borel sets [3]. Another argument in favour of the naturalness of \aleph_1 -Borel sets is that it is at least consistent that they be very well behaved. Under $MA \ \& \ 2^{\aleph_0} > \aleph_1$ they are Lebesgue measurable and have the property of Baire [5].

The \aleph_1 -Borel sets, however, are not well behaved absolutely and so the study of them is not merely an exercise in rephrasing all the known theorems about Borel sets. If $2^{\aleph_0} = \aleph_1$, for example, then all sets of reals are \aleph_1 -Borel and hence nothing specific can be said about them. Also note that the fact that the reals are second countable is very useful in the study of Borel sets, but in the study of \aleph_1 -Borel sets it must play a much smaller role.

The main result of this paper is that $2^{\aleph_0} = \aleph_1$ is not equivalent to the statement that every set of reals is \aleph_1 -Borel. This answers a question of Galvin, Prikry and Wolfson [6]. This result is found in §2. §1 contains definitions and terminology while §3 contains some results on when non- \aleph_1 -Borel sets exist.

1. Definitions and terminology. Lower case Greek letters will denote ordinals. The letter κ will be reserved for cardinals. If X is a set $\mathcal{P}(X)$ is the power set of X . Also $[X]^\kappa = \{Y \subseteq X: |Y| = \kappa\}$, $[X]^{<\kappa} = \{Y \subseteq X: |Y| < \kappa\}$ and $[X]^{<=\kappa} = \{Y \subseteq X: |Y| \leq \kappa\}$. If \mathbf{P} is a partial order and $1 \Vdash_{\mathbf{P}}$ " \mathbf{Q} is a partial order" then $\mathbf{P} * \mathbf{Q}$ denotes the iteration of \mathbf{P} followed by \mathbf{Q} .

Define by induction on the ordinals $\xi \in \omega_2$ sets $\Sigma_\xi^{\aleph_1}$, $\Pi_\xi^{\aleph_1}$ as follows:

$$(1) \Sigma_0^{\aleph_1} = \Pi_0^{\aleph_1} = \{X \subseteq \mathbb{R}: X \text{ is Borel}\};$$

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(2) for $\xi \neq 0$, $\Sigma_\xi^{\aleph_1} = \{ \cup \mathcal{Q} : \mathcal{Q} \in [\cup \{ \Pi_\eta^{\aleph_1} : \eta \in \xi \}]^{<\aleph_1} \}$;

(3) for $\xi \neq 0$, $\Pi_\xi^{\aleph_1} = \{ \mathbf{R} \setminus X : X \in \Sigma_\xi^{\aleph_1} \}$.

Clearly for $\alpha \in \beta \in \omega_2$, $\Sigma_\alpha^{\aleph_1} \subseteq \Sigma_\beta^{\aleph_1}$, $\Sigma_\alpha^{\aleph_1} \subseteq \Pi_\beta^{\aleph_1}$, $\Pi_\alpha^{\aleph_1} \subseteq \Sigma_\beta^{\aleph_1}$ and $\Pi_\alpha^{\aleph_1} \subseteq \Pi_\beta^{\aleph_1}$. Hence it is possible to define $B(\aleph_1) = \cup \{ \Pi_\xi^{\aleph_1} : \xi \in \omega_2 \} = \cup \{ \Sigma_\xi^{\aleph_1} : \xi \in \omega_2 \}$. The elements of $B(\aleph_1)$ are known as \aleph_1 -Borel sets.

2. $\text{Con}(2^{\aleph_0} > \aleph_1 \ \& \ B(\aleph_1) = \mathcal{P}(\mathbf{R}))$. As its title suggests the main goal of this section is to prove that the continuum hypothesis is not equivalent to the equality $B(\aleph_1) = \mathcal{P}(\mathbf{R})$. In fact it will be shown to be consistent with the negation of the continuum hypothesis that $\Sigma_2^{\aleph_1} = \mathcal{P}(\mathbf{R})$. Before proving this let us note that this is the best result possible.

SCHOLIUM 1. $2^{\aleph_0} = \aleph_1$ is equivalent to $\Sigma_1^{\aleph_1} \cup \Pi_1^{\aleph_1} = \mathcal{P}(\mathbf{R})$.

PROOF. One direction is trivial. For the other assume that $2^{\aleph_0} > \aleph_1$. Using the fact that $|\Sigma_0^{\aleph_1}| = 2^{\aleph_0}$ and that if $X \in \Sigma_0^{\aleph_1}$ and $|X| > \aleph_0$ then $|X| = 2^{\aleph_0}$, it is possible to inductively construct $A \subseteq \mathbf{R}$ such that for each uncountable $X \in \Sigma_0^{\aleph_1}$, $|A \cap X| = |(\mathbf{R} \setminus A) \cap X| = 2^{\aleph_0}$. Since $2^{\aleph_0} > \aleph_1$ it follows that neither A nor $\mathbf{R} \setminus A$ belongs to $\Sigma_1^{\aleph_1}$ and hence $A \in \mathcal{P}(\mathbf{R}) \setminus (\Sigma_1^{\aleph_1} \cup \Pi_1^{\aleph_1})$.

DEFINITION 2. For any $A \subseteq \mathbf{R}$ define a partial order $\mathbf{BC}(A) = (BC(A), <)$ as follows: $(f, g) \in BC(A)$ if and only if

- (1) $f: n \times n \rightarrow [[\mathbf{Q}]^2]^{<\aleph_0}$ where $n \in \omega$ and \mathbf{Q} is the set of rationals,
- (2) $g: \omega \rightarrow [A]^{<\aleph_0}$,
- (3) $|\{n \in \omega : g(n) \neq 0\}| < \aleph_0$,
- (4) $(\forall n \in \omega) (\forall (n, m) \in \text{domain}(f)) (\forall s \in g(n)) (\exists \{p, q\} \in f((n, m))) (p < s < q)$, let $(f, g) < (f', g')$ if and only if:
 - (5) $f \supseteq f'$,
 - (6) $(\forall n \in \omega) (g(n) \supseteq g'(n))$.

LEMMA 3. For any $A \subseteq \mathbf{R}$, $\mathbf{BC}(A)$ is σ -centred.

PROOF. $(\forall n \in \omega) (\forall f: n \times n \rightarrow [Q]^2) (\{(h, g) \in BC(A) : h = f\}$ is centred in $\mathbf{BC}(A)$).

If G is $\mathbf{BC}(A)$ -generic over V then let

$$G^* = \bigcup \{ f : (\exists g: \omega \rightarrow [A]^{<\aleph_0}) ((f, g) \in G) \}.$$

Standard arguments show that G^* is a function from $\omega \times \omega$ to $[[Q]^2]^{<\aleph_0}$. From G^* it is possible to define a set of reals

$$\langle G^* \rangle = \bigcup \{ \cap \{ \{s \in \mathbf{R} : (\exists \{p, q\} \in G^*((n, m))) (p < s < q)\} : m \in \omega \} n \in \omega \}.$$

Note that the set of reals $\langle G^* \rangle$ will vary from model to model.

LEMMA 4. If G is $\mathbf{BC}(A)$ -generic over V then $\langle G^* \rangle \cap V = A$. Furthermore this equality holds in any extension $M \supseteq V[G]$.

PROOF. Let $s \in A$. Then of course $s \in V$ since $A \in V$ and V is transitive. To see that $s \in \langle G^* \rangle$ note that $\{(f, g) \in BC(A) : (\exists n \in \omega) (s \in g(n))\}$ is dense in $\mathbf{BC}(A)$.

Furthermore note that if $(f, g) \in G$ and $s \in g(n)$ then $(\forall(f', g') \in G) (\forall(n, m) \in \text{domain}(f')) (\exists\{p, q\} \in f'((n, m)) (p < s < q))$. Conversely, if $s \in V \cap \mathbf{R} \setminus A$ then $\{(f, g) \in BC(A):$

$$(\exists m \in \omega) (\forall\{p, q\} \in f((n, m))) (s < \min\{p, q\} \text{ or } s > \max\{p, q\})\}$$

is dense in $BC(A)$ for each $n \in \omega$. Hence $s \notin \langle G^* \rangle$.

The last remark is clear from the absoluteness of Borel predicates [4].

DEFINITION 5. Let $BC = \Pi\{BC(A): A \subseteq \mathbf{R}\}$ and

$$wBC = \Pi\{BC(A): A \in [\mathbf{R}]^{<2^{\aleph_0}}\}$$

where the product in both cases is the Tychonoff product. It follows from Lemma 3 that both BC and wBC satisfy the countable chain condition (see [4, p. 243]).

Now define by induction on ordinals $\xi \subseteq \omega_1$ partial orders P_ξ and wP_ξ as follows:

(1) $P_0 = BC$ and $wP_0 = wBC$;

(2) for ξ a limit ordinal, P_ξ is the direct limit of $\{P_\eta: \eta \in \xi\}$ and wP_ξ is the direct limit of $\{wP_\eta: \eta \in \xi\}$;

(3) $P_{\xi+1} = P_\xi * BC$ and $wP_{\xi+1} = wP_\xi * wBC$.

Obviously both P_{ω_1} and wP_{ω_1} satisfy the countable chain condition. As usual if $\xi \in \eta$ then P_η is isomorphic to a dense subset of $P_\xi * P^{\xi, \eta}$ where $P^{\xi, \eta}$ is a name for the $\eta \setminus \xi$ -tail of the iteration as defined over an extension which is generic for P_ξ .

Also, if G is generic for P_η , then $G = G_\xi * G^{\xi, \eta}$ where G_ξ is P_ξ -generic over V and $G^{\xi, \eta}$ is $P^{\xi, \eta}$ -generic over $V[G_\xi]$. Similar remarks hold for wP_η as well. For details see [1].

LEMMA 6.

(a) If G is P_{ω_1} -generic over V , $\xi \in \omega_1$ and $V[G_\xi] \models "A \subseteq \mathbf{R}"$ then $V[G] \models "A \in \Pi_1^{\aleph_1}"$.

(b) If G is wP_{ω_1} -generic over V , $\xi \in \omega_1$ and $V[G_\xi] \models "A \in [\mathbf{R}]^{<2^{\aleph_0}}"$ then $V[G] \models "A \in \Pi_1^{\aleph_1}"$.

PROOF. Since the proofs of (a) and (b) are almost identical only (a) will be proved. First note that if $\xi \subseteq \eta \in \omega_1$ then $A \in V[G_\eta]$ and hence $V[G_{\xi+1}] = V[G_\eta][G^{\eta, \eta+1}] = V[G_\eta][H'] [H_\eta]$ where $H_\eta \times H' = G^{\eta, \eta+1}$ and H_η is $BC(A)$ -generic over $V[G_\eta]$. Of course if $\xi \subseteq \eta \in \omega_1$ then $\langle H_\eta^* \rangle \in \Sigma_0^{\aleph_1}$. It therefore suffices to show that $\bigcap \{\langle H_\eta^* \rangle: \xi \subseteq \eta \in \omega_1\} = A$ where $\langle H_\eta^* \rangle$ is interpreted in $V[G]$.

By Lemma 4 it is clear that $A \subseteq \bigcap \{\langle H_\eta^* \rangle: \xi \subseteq \eta \in \omega_1\}$. Let $V[G] \models "s \in \mathbf{R} \setminus A"$ then by the countable chain condition $s \in V[G_\eta]$ for some $\eta \in \omega_1 \setminus \xi$. But then by Lemma 4 $\langle H_\eta^* \rangle \cap V[G_\eta] = A$ and hence $s \notin \langle H_\eta^* \rangle$.

LEMMA 7.

(a) If G is P_{ω_1} -generic over V and $A \subseteq \mathbf{R} \cap V[G_\xi]$ for some $\xi \in \omega_1$ and $A \in V[G]$ then $V[G] \models "A = \bigcup \{A_\zeta: \xi \subseteq \zeta \in \omega_1\}"$ where $A_\zeta \in V[G_\zeta]$ for $\xi \subseteq \zeta \in \omega_1$.

(b) If G is wP_{ω_1} -generic over V where $V \models "2^{\aleph_0} = 2^{2^{\aleph_0}}"$ and $A \subseteq \mathbf{R} \cap V[G_\xi]$ for some $\xi \in \omega_1$ and $V[G] \models "A \in [\mathbf{R}]^{<2^{\aleph_0}}"$ then $V[G] \models "A = \bigcup \{A_\zeta: \xi \subseteq \zeta \in \omega_1\}"$ where $V[G_\zeta] \models "A_\zeta \in [\mathbf{R}]^{<2^{\aleph_0}}"$ for $\xi \subseteq \zeta \in \omega_1$.

PROOF. Only (b) will be proved since the proof of (a) is similar and even easier. It is easy to see that if $W \models "2^{\aleph_0} = 2^{2^{\aleph_0}}"$ then $|wBC| = 2^{\aleph_0}$ and hence if H is wBC over W then $W[H] = "2^{\aleph_0} = (2^{\aleph_0})^W = (2^{2^{\aleph_0}})^W = 2^{2^{\aleph_0}}"$. Hence $V[G] = "2^{\aleph_0} = (2^{\aleph_0})^V"$.

Now let $V[G] = V[G_\xi][G^{\xi, \omega_1}]$ and choose $\mathcal{A} \in V[G_\xi]$ a P^{ξ, ω_1} -name for A . We may assume that \mathcal{A} is nice (i.e. $\mathcal{A} = \bigcup \{ \check{s} \times A(s) : s \in \mathbf{R} \cap V[G_\xi] \}$ where each $A(s)$ is an antichain). Define, in $V[G_\xi]$,

$$\mathcal{A}_\xi = \bigcup \{ \check{s} \times A(s) : s \in \mathbf{R} \text{ and } A(s) \subseteq \mathbf{P}^{\xi, \check{s}} \}$$

for $\xi \subseteq \zeta \in \omega_1$. Let A_ζ be the interpretation in $V[G]$ of \mathcal{A}_ζ . Then $A_\zeta \subseteq A$ and hence $|A_\zeta| < 2^{\aleph_0}$. But $A_\zeta \in V[G_\zeta][G^{\xi, \zeta}] = V[G_\zeta]$ and, since $(2^{\aleph_0})^{V[G_\zeta]} = (2^{\aleph_0})^{V[G]}$, $V[G_\zeta] \models "A_\zeta \in [\mathbf{R}]^{< 2^{\aleph_0}}"$. To see that $V[G] \models "A = \bigcup \{ A_\zeta : \xi \subseteq \zeta \in \omega_1 \}"$, use the countable chain condition.

LEMMA 8.

(a) If G is \mathbf{P}_{ω_1} -generic over V and $A \in V[G]$ then $A = \bigcup \{ A_\xi : \xi \in \omega_1 \}$ where $(\forall \xi \in \omega_1) (\exists \eta \in \omega_1) (A_\xi \in V[G_\eta])$.

(b) If G is $w\mathbf{P}_{\omega_1}$ -generic over V and $V \models "2^{\aleph_0} = 2^{2^{\aleph_0}}"$ and $A \in V[G]$ then $A = \bigcup \{ A_\xi : \xi \in \omega_1 \}$ where

$$(\forall \xi \in \omega_1) (\exists \eta \in \omega_1) (V[G_\eta] \models "A_\xi \in [\mathbf{R}]^{< 2^{\aleph_0}}").$$

PROOF. Use the countable chain condition and Lemma 7.

THEOREM 9.

(a) $\text{Con}(\text{ZF}) \Rightarrow \text{Con}(\text{ZFC} \ \& \ 2^{\aleph_0} > \aleph_1 \ \& \ \Sigma_2^{\aleph_1} = \mathcal{P}(\mathbf{R}))$

(b) If $\text{cf}(\kappa) > \omega$ then $\text{Con}(\text{ZF}) \Rightarrow \text{Con}(\text{ZFC} \ \& \ 2^{\aleph_0} = \kappa \ \& \ \Sigma_2^{\aleph_1} \supseteq [\mathbf{R}]^{< 2^{\aleph_0}})$.

PROOF. Use Lemma 6 and Lemma 8.

SCHOLIUM 10. In the model for Theorem 9 (a), $2^{\aleph_0} = (\aleph_1)^V$.

PROOF. It is easy to prove by induction that if G is \mathbf{P}_{ω_1} -generic over V and $\xi \in \omega_1$, $V[G_\xi] \models 2^{\aleph_0} \geq (\aleph_\xi)^V$ and since $|\mathbf{P}_{\omega_1}| = \aleph_{\omega_1}$, the result follows.

A natural question which arises is whether or not it is possible to have $\text{cf}(2^{\aleph_0}) > \omega_1$ and $B(\aleph_1) = \mathcal{P}(\mathbf{R})$. In the next section we will indicate at least one difficulty which must be overcome if a model of $\text{cf}(2^{\aleph_0}) > \omega_1$ and $B(\aleph_1) = \mathcal{P}(\mathbf{R})$ is to be obtained. Also, since $2^{\aleph_0} = \aleph_1 \Rightarrow B(\aleph_1) = \mathcal{P}(\mathbf{R})$ and Theorem 9(a) shows that $B(\aleph_1) = \mathcal{P}(\mathbf{R})$ is consistent with $\text{cf}(2^{\aleph_0}) = \omega_1 < 2^{\aleph_0}$, it might be conjectured that $\text{cf}(2^{\aleph_0}) = \omega_1 \Rightarrow B(\aleph_1) = \mathcal{P}(\mathbf{R})$. In the next section it will be shown that this is not the case.

Finally we remark that J. Baumgartner has constructed a different model for the proof of Theorem 9(a). He starts with a model of GCH and constructs an iteration of length ω_1 such that at each successor stage $\xi + 1$, a model of MA & $2^{\aleph_0} = \aleph_{\xi+1}$ is obtained. The fact that under MA every set of reals of size $< 2^{\aleph_0}$ is a Q -set [8] is used to show that $\Sigma_2^{\aleph_1} = \mathcal{P}(\mathbf{R})$. In fact even more is true: Every set of reals is of the form $\bigcup \{ \bigcap \{ \bigcup \{ A_{\xi, \eta, \zeta} : \xi \in \omega_1 \} : \eta \in \omega_1 \} : \zeta \in \omega_1 \}$, where the sets $A_{\xi, \eta, \zeta}$ are not only Borel but in fact closed. The referee has pointed out that the same model is used in [2] to obtain results on the π -character of βN .

3. Models with non- \aleph_1 -Borel sets. We begin this section by recalling the following classical theorem of Hausdorff. Every uncountable Borel set has cardinality 2^{\aleph_0} ([4, p. 507]). The analogue of this statement for \aleph_1 -Borel sets is: Every \aleph_1 -Borel set of cardinality greater than \aleph_1 has cardinality 2^{\aleph_0} . The results of the previous section show that this statement is not a theorem of ZFC. However it is easy to construct models where 2^{\aleph_0} is arbitrarily large and this statement does hold.

Recall that ordinary Borel sets can be coded by functions $c: \omega \rightarrow \omega$ [4]. By similar arguments it can be shown that \aleph_1 -Borel sets can be coded by functions $c: \omega_1 \rightarrow \omega_1$. If $c \in V$ and $V \models "c: \omega_1 \rightarrow \omega_1"$ then the \aleph_1 -Borel set coded by c in V will be denoted by $c(V)$. It is easy to verify that if V and V' are transitive models of ZFC and $\omega_1^V = \omega_1^{V'}$ and $\{c, r\} \subseteq V \cap V'$, where $r \in \mathbf{R}$ and $c \in {}^{\omega_1}\omega_1$, then $V \models "r \in c(V)"$ if and only if $V' \models "r \in c(V)"$. This fact will be used in the results below. However, the reader is cautioned against concluding that all of the absoluteness results which hold for ordinary Borel sets also hold for \aleph_1 -Borel sets. The following observation illustrates this phenomenon.

SCHOLIUM 11. *There is a generic extension of V , $V[G]$, and an \aleph_1 -Borel code $c \in {}^{\omega_1}\omega_1 \cap V$ such that $V \models "c(V) = 0"$ but $V[G] \models "c(V[G]) \neq 0"$. Recall that this is impossible for Borel codes.*

PROOF. The result will be shown for 2^ω rather than \mathbf{R} . Let $\{a_\xi: \xi \in \omega_1\} \cup \{b_\xi: \xi \in \omega_1\} \subseteq 2^\omega$ be a Hausdorff gap. Then, as in [3], define $F_{\alpha\delta}$'s $\{A_\xi: \xi \in \omega_1\}$ such that $f \in A_\xi$ if and only if $a_\xi \leq f < b_\xi$ (mod finite). Let $c \in {}^{\omega_1}\omega_1$ be a code for $\bigcap \{A_\xi: \xi \in \omega_1\}$. Clearly $V \models "c(V) = 0"$. Now let $V[G]$ be the generic extension obtained by collapsing ω_1 . Then $V[G] \models "c(V[G]) \neq 0"$.

THEOREM 12. *If $V \models "2^{\aleph_0} = \aleph_1"$ & \mathbf{P} is the product of κ many Cohen partial orders then in any generic extension by \mathbf{P} every set of reals of size greater than \aleph_1 has size 2^{\aleph_0} .*

PROOF. Let $\mathbf{P} = \prod \{C_\xi: \xi \in \kappa\}$ where each C_ξ is countable and let G be \mathbf{P} -generic. For $\Gamma \subseteq \kappa$ let G_Γ be generic over $\prod \{C_\xi: \xi \in \Gamma\}$ in such a way that for each Γ , $G_\Gamma \times G_{\kappa \setminus \Gamma} = G$.

Now suppose that in $V[G]$, $A \subseteq \mathbf{R}$ is \aleph_1 -Borel. Let $c: \omega_1 \rightarrow \omega_1$ be an \aleph_1 -Borel code for A . Choose $\Gamma \in [\kappa]^{\aleph_1}$ such that $c \in V[G_\Gamma]$.

If $A \subseteq V[G_\Gamma]$ then $|A| < \aleph_1$. Otherwise choose $s \in A \setminus V[G_\Gamma]$. Then there is a countable $\Lambda \subseteq \kappa$ such that $s \in V[G_\Gamma][G_\Lambda]$. Since $\prod \{C_\xi: \xi \in \Lambda\}$ is isomorphic to the Cohen partial order \mathbf{C} , we have for some $p \in \mathbf{C}$

$$V[G_\Gamma] \models "p \Vdash \check{c}(V[G_\Gamma][H]) \setminus \check{c}(V[G_\Gamma]) \neq 0"$$

where H is a name for the generic set. But clearly $|\{\xi \in \kappa \setminus \Gamma: p \in G_{\{\xi\}}\}| = \kappa$ and so $|c(V[G])| = \kappa$.

COROLLARY 13.

$$2^{2^{\aleph_0}} = 2^{\aleph_1} \text{ does not imply } B(\aleph_1) = \mathcal{P}(\mathbf{R}).$$

(Note that $B(\aleph_1) = \mathcal{P}(\mathbf{R})$ does imply $2^{2^{\aleph_0}} = 2^{\aleph_1}$.)

COROLLARY 14.

$\text{cf}(2^{\aleph_0}) = \omega_1$ does not imply $B(\aleph_1) = \mathcal{P}(\mathbf{R})$.

Let us now consider the problem posed in the last section: Is it possible to have $\text{cf}(2^{\aleph_0}) > \omega_1$ & $B(\aleph_1) = \mathcal{P}(\mathbf{R})$ simultaneously? The next result shows that none of the standard ways of adding generic reals will provide a positive answer to this question.

SCHOLIUM 15. Suppose that $\{V_\xi: \xi \in \kappa\}$ is an increasing sequence of models of ZFC such that

(1) $(\forall \xi \in \eta \in \kappa) (\mathbf{R} \cap (V_\eta \setminus V_\xi) \neq \emptyset)$,

(2) ${}^{\omega_1}\omega_1 \subseteq \bigcup \{V_\xi, \xi \in \kappa\}$.

Then $B(\aleph_1) \neq \mathcal{P}(\mathbf{R})$.

PROOF. First choose for $\xi \in \kappa$, $a_\xi \in \mathbf{R} \cap (V_{\xi+1} \setminus V_\xi)$. Choose a canonical bijection $\Phi: \mathbf{R} \rightarrow \mathcal{P}(\mathbf{Q})$. Let $X = \{a_\xi + s: \xi \in \kappa \text{ \& } s \in \Phi(a_{\xi+1})\}$. To see that X is not \aleph_1 -Borel suppose $c: \omega_1 \rightarrow \omega_1$ is an \aleph_1 -Borel code for X . By (2) choose $\xi \in \kappa$ such that $c \in V_{\xi+1}$. Then $c(V_{\xi+1}) \cap (a_\xi + \mathbf{Q}) \in V_{\xi+1}$. But clearly $\{a_\xi + \mathbf{Q}: \xi \in \kappa\}$ is pairwise disjoint. Hence $c(V_{\xi+1}) \cap (a_\xi + \mathbf{Q}) = X \cap (a_\xi + \mathbf{Q}) = \{a_\xi + s: s \in \Phi(a_{\xi+1})\}$. But then $a_{\xi+1} \in V_{\xi+1}$.

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