ON THE DIVISIBLE PART OF
THE BRAUER GROUP OF A FIELD

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Abstract. For a field $k$ and an odd prime $p \neq \text{char}(k)$ such that the $p$-primary component $B(k)_{(p)}$ of the Brauer group $B(k)$ of $k$ is not zero there exists a finite extension $k'/k$ such that $B(k'_{(p)})$ contains a nontrivial divisible subgroup.

Let $k$ be an arbitrary field, $p \neq \text{char}(k)$ a prime, and $B(k)_{(p)}$ the $p$-primary component of the Brauer group $B(k)$ of $k$. Brumer and Rosen [1] conjecture that either $2B(k)_{(p)} = 0$ or $B(k)_{(p)}$ contains a nontrivial divisible subgroup. As an easy consequence of our investigation of the relative Brauer group of a maximal $p$-extension [3], we are able to show the conjecture is true modulo a finite extension of $k$. For the facts about profinite groups used here, we refer the reader to [2].

Theorem. Let $k$ be a field and $p \neq \text{char}(k)$ a prime. If $2B(k)_{(p)}$ is not zero, then there exists a finite separable extension $k'/k$ such that $B(k'_{(p)})$ contains a nontrivial divisible subgroup and the maximal power of $p$ dividing $[k : k']$ is at most 2.

We need a simple lemma for the proof. For $V$ a profinite group let $V_{(p)}$ denote the smallest normal subgroup such that $V/V_{(p)}$ is a pro-$p$-group. Then $F_{(p)} < W_{(p)}$ for $V < W$.

Lemma. Let $G$ be a profinite group, $S$ a pro-$p$-subgroup, and $\mathcal{V}$ the set of open subgroups of $G$ containing $S$. Then $S = \lim \sup_{\mathcal{V}} V_{(p)}$.

Proof. Consider the exact sequence $1 \to V_{(p)} \to V \to V/V_{(p)} \to 1$ for $V \in \mathcal{V}$. Since $\cap_{V \in \mathcal{V}} V = S$ and the inverse limit is exact for profinite groups, we have only to show that $\cap_{V \in \mathcal{V}} V_{(p)} = 1$. Let $U$ be an open normal subgroup of $G$. Then $V = SU$ is in $\mathcal{V}$ and $V_{(p)} < U$ because $V/U$ is pro-$p$. So $\cap_{V \in \mathcal{V}} V_{(p)}$ is contained in the intersection of all open normal subgroups of $G$ which is trivial.

Proof of the Theorem. Let $k_s$ be the separable closure of $k$, with Galois group $G$ over $k$, $\mu_{p^n}$ the group of $p^n$th roots of unity in $k_s$, and $\mu = \cup_{n=1,2,\ldots} \mu_{p^n}$. Denote by $k_0$ the field $k(\mu_p)$ if $p \neq 2$, and $k(\mu_2)$ if $p = 2$. Since its degree $[k_0 : k]$ divides $p - 1$ or 2, respectively, we have $B(k_0)_{(p)} \neq 0$. Hence it suffices to consider the situation $k = k_0$ and $B(k)_{(p)} \neq 0$ and to show there exists a finite extension $k'$ of $k$ in $k_s$, of degree $[k' : k]$ prime to $p$, such that $B(k'_{(p)})$ contains a nontrivial divisible subgroup.
By [1, Lemma 2] we have $B(k)(p) = H^2(G, \mu)$. Let $S$ be a Sylow $p$-subgroup of $G$. Since $cd_p(G) = cd(S)$, $S$ cannot be a free pro-$p$-group and hence, by the lemma, there exists an open subgroup $V$ of $G$ containing $S$, such that the group $V/V_{(p)}$ is not pro-$p$-free. Let $\tilde{k}$ and $K$ be the subfields of $k$, left fixed by the groups $V$ and $V_{(p)}$, respectively. Then $[\tilde{k} : k] = [G : V]$ is prime to $p$ and $\tilde{k}$ contains the $p$th or fourth roots of unity. The field $K$ is the maximal $p$-extension of $\tilde{k}$ and the Galois group $\text{Gal}(K/\tilde{k})$ is $V/V_{(p)}$, which is not pro-$p$-free. Hence, by [3, Satz 3(a)], the part $B(K/\tilde{k})$ of the Brauer group $B(\tilde{k})$ split by $K$ contains a nontrivial divisible $(p$-primary) subgroup.

REFERENCES


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