A SIMPLE PROOF OF THE EXTENSION THEOREM OF SEQUENCES OF DIVIDED POWERS IN CHARACTERISTIC \( p \)

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Abstract. Using the idea of relative Hopf modules, a short proof of the extension theorem of sequences of divided powers in irreducible cocommutative Hopf algebras over a field of characteristic \( p > 0 \) is presented.

Let \( k \) be a field of characteristic \( p > 0 \). Let \( H \) be an irreducible cocommutative Hopf algebra over \( k \). The \( V \)-map for \( H \) [1, (4.1)]

\[ V : H \to k^{1/p} \otimes H \]

is a \( \frac{1}{p} \)-linear Hopf algebra map [1, p. 279] with kernel \( HL \) where \( L = P(H) \), the primitives in \( H \) [4, Theorem 1]. We define a descending set of Hopf subalgebras \( \{ V^n(H) \}_{n \geq 0} \) inductively as follows: \( V^0(H) = H, \ V^n(H) = V(V^{n-1}(H)) \cap H. \ (V^1(H) = V(H) \cap H \) is different from \( V(H) \). Since \( V(H) \) is a \( k^{1/p} \)-Hopf subalgebra of \( k^{1/p} \otimes H \), it is easy to check that each \( V^n(H) \) is a \( k \)-Hopf subalgebra of \( H \).

An element \( x \in H \) has coheight \( n \) if \( x \in V^n(H) \). For each integer \( e > 0 \), the integer \( \| e \| > 0 \) is defined by

\[ p^{\| e \|} \leq e < p^{\| e \| + 1}. \]

A set of elements \( x_0 = 1, x_1, \ldots, x_n \) (\( n \) finite) in \( H \) is called an \( n \)-sequence of divided powers if

\[ \Delta(x_i) = \sum_{j=0}^{i} x_j \otimes x_{i-j}, \quad 0 < i < n. \]

**Theorem A** [4, Lemma 7; and 2, Theorem 2]. Let \( t < p^{n+1} \) and let \( x_0, x_1, \ldots, x_{t-1} \) be a sequence of divided powers in \( H \) where \( x_i \) has coheight \( n - \| i \|, 0 < i < t \). There is an element \( x_i \) in \( H \) of coheight \( n - \| t \| \) such that \( x_0, x_1, \ldots, x_{t-1}, x_i \) is a sequence of divided powers.

The following extension theorem of sequences of divided powers is a key lemma to determine the coalgebra structure of \( H \) [4, Theorems 2 and 3]. The original proof of Sweedler, which consists of several steps, is done by induction on \( n \) and \( t \). In the following, we give an alternative proof, where we do not use induction, but the idea of relative Hopf modules [5] instead.

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Proof. Replacing $H$ by $V^{|\ell|}(H)$, we may assume $n = |\ell|$. Let $\tilde{H}$ be the free $k$-algebra generated by $H$ and one indeterminate $z$. Thus, if $A$ is a $k$-algebra and $\varphi: H \to A$ an algebra map, then for any $a \in A$, there is a unique algebra map $\tilde{\varphi}: \tilde{H} \to A$ such that $\tilde{\varphi}|H = \varphi$ and $\tilde{\varphi}(z) = a$. Using this universal mapping property, define algebra maps

$$\tilde{\Delta}: \tilde{H} \to \tilde{H} \otimes \tilde{H}, \quad \tilde{\varepsilon}: \tilde{H} \to k$$

by the rule:

$$\tilde{\Delta}|H = \Delta \quad \text{(comultiplication of } H), \quad \tilde{\Delta}(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{n-1} x_i \otimes x_{i+1}, \quad \tilde{\varepsilon}|H = \varepsilon \quad \text{(augmentation of } H), \quad \tilde{\varepsilon}(z) = 0. \quad \text{Then } (\tilde{H}, \tilde{\Delta}, \tilde{\varepsilon}) \text{ is an irreducible cocommutative Hopf algebra containing } H \text{ as a Hopf subalgebra. Since } x_0, x_1, \ldots, x_{n-1}, z \text{ is a } t \text{-sequence of divided powers in } \tilde{H}, \quad V(z) = 0 \text{ if } p | t \text{ and } \quad V(z) = x_t \text{ if } t = ps. \quad \text{In the latter case, } x_t \text{ has coheight } |t| - |s| = 1. \quad \text{Hence } \quad V(z) \in V(H).$$

Since $V$ is a semilinear Hopf algebra map, it follows that $V(\tilde{H}) = V(H)$. Let $\tilde{L} = P(\tilde{H})$, the primitives in $\tilde{H}$. Let $U$ (resp. $\tilde{U}$) be the restricted universal enveloping algebra of $L$ (resp. $\tilde{L}$). Then $U$ (resp. $\tilde{U}$) is a Hopf subalgebra of $H$ (resp. $\tilde{H}$) [3, Proposition 13.2.3]. We claim that the multiplication in $\tilde{H}$ induces an isomorphism

$$H \otimes_U \tilde{U} \iso \tilde{H}.$$ 

Indeed, both sides are right $(\tilde{H}, \tilde{U})$-Hopf modules [5, p. 454] and the map is a homomorphism. Since $\tilde{H}$ is irreducible, $\tilde{H}$ is a free left (or right) $\tilde{U}$-module [5, Proposition 3]. Hence the category of right $(\tilde{H}, \tilde{U})$-Hopf modules is equivalent to the category of right $\tilde{H} / \tilde{H}L$-comodules [5, Theorem 1], where the equivalence is given by $M \mapsto M / M \tilde{L}$. If we apply this equivalence functor to the above homomorphism, we get the canonical map $H / HL \to \tilde{H} / \tilde{H}L$ which is an isomorphism, since $H / HL \iso V(H), \quad \tilde{H} / \tilde{H}L \iso V(\tilde{H})$ and $V(\tilde{H}) = V(H)$. This proves our claim.

Let $X$ be a basis of $\tilde{L}$ modulo $L$. Let $\Lambda$ be the set of all functions from $X$ to $\{0, 1, \ldots, p - 1\}$ with finite support. Give a total order on $X$. For each $f$ in $\Lambda$, put

$$[f] = \frac{c_1^{e_1} \cdots c_n^{e_n}}{e_1! \cdots e_n!} \quad \text{and} \quad |f| = e_1 + \cdots + e_n$$

where $\{c_1, \ldots, c_n\}$ is the support of $f$ with $c_1 < \cdots < c_n$ and $e_i = f(c_i)$. Then $\{[f] | f \in \Lambda\}$ is a free basis of the left $U$-module $\tilde{U}$ (Poincaré-Birkhoff-Witt), hence of the left $H$-module $\tilde{H}$, and we have

$$\Delta[f] = \sum_{f = g + h} [g] \otimes [h].$$

Write $z = \sum_{f \in \Lambda} z[f]$, $z_f \in H$. Then

$$\tilde{\Delta}(z) = \sum \tilde{\Delta}(z_{g+h})([g] \otimes [h])$$

where the sum is taken over the set of all $g, h \in \Lambda$ with $g + h \in \Lambda$. Since $\tilde{\Delta}(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{n-1} x_i \otimes x_{i+1}$, and $\{[g] \otimes [h] | g, h \in \Lambda\}$ is a free basis of the left $H \otimes H$-module $\tilde{H} \otimes \tilde{H}$, it follows from comparison of the coefficients that $z_f = 0$ for $|f| > 1$ and $z_f \in k$ for $|f| = 1$. Put $x_t = z - \sum_{|f|=1} z_f[f]$. Then $x_t \in H$ and $\tilde{\Delta}(z) - z \otimes 1 - 1 \otimes z = \Delta(x_t) - x_t \otimes 1 - 1 \otimes x_t$. Hence $x_0, x_1, \ldots, x_{n-1}, x_t$ is a sequence of divided powers in $H$. Q.E.D.
The above idea of proof yields more general results. Note that we merely used the fact that \( V(H) = V(\tilde{H}) \) in the latter part of the above proof. Hence, what we proved actually is the following

**Theorem B.** Let \( \tilde{H} \) be an irreducible cocommutative Hopf algebra and let \( H \subset \tilde{H} \) be a Hopf subalgebra. Assume \( V(H) = V(\tilde{H}) \). If \( z \in \tilde{H} \) satisfies

\[
\Delta(z) = z \otimes 1 - 1 \otimes z \in H \otimes H
\]

there is an element \( x \in H \) such that

\[
\Delta(x) = x \otimes 1 - 1 \otimes x = \Delta(z) - z \otimes 1 - 1 \otimes z.
\]

It is enough to assume \( V(z) \in V(H) \) instead of \( V(H) = V(\tilde{H}) \). (Replace \( \tilde{H} \) by the Hopf subalgebra generated by \( H \) and \( z \).)

The above theorem can be interpreted as a cohomological vanishing theorem of the underlying coalgebras of irreducible cocommutative Hopf algebras. To clarify the meaning, for a pointed irreducible cocommutative coalgebra \( C \), let \( C^+ = \text{Ker}(\varepsilon) \) and

\[
\delta: C^+ \to C^+ \otimes C^+, \quad \delta(x) = \Delta(x) - x \otimes 1 - 1 \otimes x
\]

where 1 denotes the unique group-like element of \( C \). We want to determine the image \( \delta(C^+) \). Let \( \delta_n: C^+ \to \otimes^{n+1} C^+ \) be the \( n \) times iterated \( \delta \)-map. Let

\[
u = \sum_i x_i \otimes y_i \in C^+ \otimes C^+
\]

be an element satisfying

(a) \( \sum_i x_i \otimes y_i = \sum_i y_i \otimes x_i \),

(b) \( \sum_i \delta(x_i) \otimes y_i = \sum_i x_i \otimes \delta(y_i) \).

There is a pointed irreducible cocommutative coalgebra \( C' = C \otimes k \) which contains \( C \) as a subcoalgebra and satisfies

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + u, \quad \varepsilon(z) = 0.
\]

Then \( V(z) \) is determined by \( u \) as follows: \( \sum_i \delta_{p-2}(x_i) \otimes y_i \) is a symmetric tensor in \( \otimes^p C^+ \). Let

\[
v: (\text{the symmetric tensors in } \otimes^p C^+) \to k^{1/p} \otimes C^+
\]

be the \( \frac{1}{p} \)-linear map defined [1, Theorem 4.1.1(a), p. 273] (where denoted by \( V \)). Put \( v(u) = v(\sum_i \delta_{p-2}(x_i) \otimes y_i) \). Then \( V(z) \) is precisely \( v(u) \).

If \( C \) underlies a Hopf algebra, then the image \( \delta(C^+) \) can be characterized as follows.

**Theorem C.** Let \( H \) be an irreducible cocommutative Hopf algebra. The image \( \delta(H^+) \) is precisely the set of elements \( u \) in \( H^+ \otimes H^+ \) satisfying (a), (b) and (c)

\[
v(u) \in V(H).
\]

**Proof.** If \( u = \delta(x) \) with \( x \in H^+ \), then \( u \) satisfies (a), (b) and \( v(u) = V(x) \in V(H) \). Conversely, if \( u \) satisfies (a), (b), (c), let \( \tilde{H} \) be the Hopf algebra generated by \( H \) and one indeterminate \( z \) with \( \delta(z) = u, \varepsilon(z) = 0 \). It follows from \( V(z) = v(u) \in V(H) \) that \( V(H) = V(\tilde{H}) \). Hence \( \delta(z) = \delta(x) \) for some \( x \in H \) by Theorem B. Q.E.D.
Theorem A follows from Theorem C applied to $u = \sum_{i=1}^{t-1} x_i \otimes x_{t-i}$ and $\mathcal{V}^{n-\|\mathcal{E}\|}(H)$ as $H$.

**References**


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