A SIMPLE PROOF OF THE EXTENSION THEOREM OF SEQUENCES OF DIVIDED POWERS IN CHARACTERISTIC $p$

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Abstract. Using the idea of relative Hopf modules, a short proof of the extension theorem of sequences of divided powers in irreducible cocommutative Hopf algebras over a field of characteristic $p > 0$ is presented.

Let $k$ be a field of characteristic $p > 0$. Let $H$ be an irreducible cocommutative Hopf algebra over $k$. The $V$-map for $H$ (1, (4.1))

$$V: H \rightarrow k^{1/p} \otimes H$$

is a $k^{1/p}$-linear Hopf algebra map [1, p. 279] with kernel $L = P(H)$, the primitives in $H$ [4, Theorem 1]. We define a descending set of Hopf subalgebras $\{ V^n(H) \}_{n \geq 0}$ inductively as follows: $V^0(H) = H$, $V^n(H) = V(V^{n-1}(H)) \cap H$. $(V^1(H) = V(H) \cap H$ is different from $V(H)$.) Since $V(H)$ is a $k^{1/p}$-Hopf subalgebra of $k^{1/p} \otimes H$, it is easy to check that each $V^n(H)$ is a $k$-Hopf subalgebra of $H$. An element $x \in H$ has coheight $n$ if $x \in V^n(H)$. For each integer $e > 0$, the integer $\|e\| > 0$ is defined by

$$p^{\|e\|} \leq e < p^{\|e\|+1}.$$ 

A set of elements $x_0 = 1, x_1, \ldots, x_n$ ($n$ finite) in $H$ is called an $n$-sequence of divided powers if

$$\Delta(x_i) = \sum_{j=0}^{i} x_j \otimes x_{i-j}, \quad 0 < i < n.$$ 

Theorem A [4, Lemma 7; and 2, Theorem 2]. Let $t < p^{n+1}$ and let $x_0, x_1, \ldots, x_{t-1}$ be a sequence of divided powers in $H$ where $x_i$ has coheight $n - \|i\|$, $0 < i < t$. There is an element $x_i$ in $H$ of coheight $n - \|t\|$ such that $x_0, x_1, \ldots, x_{t-1}, x_i$ is a sequence of divided powers.

The following extension theorem of sequences of divided powers is a key lemma to determine the coalgebra structure of $H$ [4, Theorems 2 and 3].

The original proof of Sweedler, which consists of several steps, is done by induction on $n$ and $t$. In the following, we give an alternative proof, where we do not use induction, but the idea of relative Hopf modules [5] instead.

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PROOF. Replacing $H$ by $V^{|s|}(H)$, we may assume $n = |s|$. Let $\tilde{H}$ be the free $k$-algebra generated by $H$ and one indeterminate $z$. Thus, if $A$ is a $k$-algebra and $\varphi : H \rightarrow A$ an algebra map, then for any $a \in A$, there is a unique algebra map $\tilde{\varphi} : \tilde{H} \rightarrow A$ such that $\tilde{\varphi}|H = \varphi$ and $\tilde{\varphi}(z) = a$. Using this universal mapping property, define algebra maps

$$\tilde{\Delta} : \tilde{H} \rightarrow \tilde{H} \otimes \tilde{H}, \quad \tilde{\varepsilon} : \tilde{H} \rightarrow k$$

by the rule: $\tilde{\Delta}|H = \Delta$ (comultiplication of $H$), $\tilde{\Delta}(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{r-1} x_i \otimes x_{r-i}$, $\tilde{\varepsilon}|H = \varepsilon$ (augmentation of $H$), $\tilde{\varepsilon}(z) = 0$. Then $(\tilde{H}, \tilde{\Delta}, \tilde{\varepsilon})$ is an irreducible cocommutative Hopf algebra containing $H$ as a Hopf subalgebra. Since $x_0, x_1, \ldots, x_{r-1}, z$ is a $t$-sequence of divided powers in $\tilde{H}$, $V(z) = 0$ if $p \nmid t$ and $V(z) = x_j$ if $t = ps$. In the latter case, $x_j$ has coheight $|\ell| - |s| = 1$. Hence $V(z) \in V(H)$. Since $V$ is a semilinear Hopf algebra map, it follows that $V(\tilde{H}) = V(H)$. Let $\tilde{L} = P(\tilde{H})$, the primitives in $\tilde{H}$. Let $U$ (resp. $\tilde{U}$) be the restricted universal enveloping algebra of $L$ (resp. $\tilde{L}$). Then $U$ (resp. $\tilde{U}$) is a Hopf subalgebra of $H$ (resp. $\tilde{H}$) [3, Proposition 13.2.3]. We claim that the multiplication in $\tilde{H}$ induces an isomorphism

$$H \otimes_U \tilde{U} \cong \tilde{H}.$$

Indeed, both sides are right $(\tilde{H}, \tilde{U})$-Hopf modules [5, p. 454] and the map is a homomorphism. Since $\tilde{H}$ is irreducible, $\tilde{H}$ is a free left (or right) $\tilde{U}$-module [5, Proposition 3]. Hence the category of right $(\tilde{H}, \tilde{U})$-Hopf modules is equivalent to the category of right $\tilde{H} / \tilde{H}L$-comodules [5, Theorem 1], where the equivalence is given by $M \mapsto M / M\tilde{L}$. If we apply this equivalence functor to the above homomorphism, we get the canonical map $H / HL \rightarrow \tilde{H} / \tilde{H}L$ which is an isomorphism, since $H / HL \simeq V(H)$, $\tilde{H} / \tilde{H}L \simeq V(\tilde{H})$ and $V(\tilde{H}) = V(H)$. This proves our claim.

Let $X$ be a basis of $\tilde{L}$ modulo $L$. Let $\Lambda$ be the set of all functions from $X$ to $\{0, 1, \ldots, p - 1\}$ with finite support. Give a total order on $X$. For each $f$ in $\Lambda$, put

$$[f] = \frac{c_1 e_1 \cdots c_n e_n}{e_1! \cdots e_n!} \quad \text{and} \quad |f| = e_1 + \cdots + e_n$$

where $\{c_1, \ldots, c_n\}$ is the support of $f$ with $c_1 < \cdots < c_n$ and $e_i = f(c_i)$. Then $\{[f] : f \in \Lambda\}$ is a free basis of the left $U$-module $U$ (Poincaré-Birkhoff-Witt), hence of the left $H$-module $\tilde{H}$, and we have

$$\Delta[f] = \sum_{g+h} [g] \otimes [h].$$

Write $z = \sum_{f \in \Lambda} z_f[f]$, $z_f \in H$. Then

$$\tilde{\Delta}(z) = \sum \tilde{\Delta}(z_{g+h})([g] \otimes [h])$$

where the sum is taken over the set of all $g, h \in \Lambda$ with $g + h \in \Lambda$. Since $\tilde{\Delta}(z) = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{r-1} x_i \otimes x_{r-i}$, and $\{[g] \otimes [h] : g, h \in \Lambda\}$ is a free basis of the left $H \otimes H$-module $H \otimes \tilde{H}$, it follows from comparison of the coefficients that $z_f = 0$ for $|f| > 1$ and $z_f \in k$ for $|f| = 1$. Put $x_i = z - \sum_{j=1}^{|s|} z_f[f]$. Then $x_i \in H$ and $\tilde{\Delta}(z) - z \otimes 1 + 1 \otimes z = \Delta(x_i) - x_i \otimes 1 + 1 \otimes x_i$. Hence $x_0, x_1, \ldots, x_{r-1}, x_i$ is a sequence of divided powers in $H$. Q.E.D.
The above idea of proof yields more general results. Note that we merely used the fact that $V(H) = V(\tilde{H})$ in the latter part of the above proof. Hence, what we proved actually is the following

**Theorem B.** Let $\tilde{H}$ be an irreducible cocommutative Hopf algebra and let $H \subset \tilde{H}$ be a Hopf subalgebra. Assume $V(H) = V(\tilde{H})$. If $z \in \tilde{H}$ satisfies

$\Delta(z) - z \otimes 1 - 1 \otimes z \in H \otimes H$

there is an element $x \in H$ such that

$\Delta(z) - z \otimes 1 - 1 \otimes z = \Delta(x) - x \otimes 1 - 1 \otimes x$.

It is enough to assume $V(z) \in V(H)$ instead of $V(H) = V(\tilde{H})$. (Replace $\tilde{H}$ by the Hopf subalgebra generated by $H$ and $z$.)

The above theorem can be interpreted as a cohomological vanishing theorem of the underlying coalgebras of irreducible cocommutative Hopf algebras. To clarify the meaning, for a pointed irreducible cocommutative coalgebra $C$, let $C^+ = \text{Ker}(e)$ and

$\delta: C^+ \to C^+ \otimes C^+,$ $\delta(x) = \Delta(x) - x \otimes 1 - 1 \otimes x$

where 1 denotes the unique group-like element of $C$. We want to determine the image $\delta(C^+)$. Let $\delta_n: C^+ \to \otimes^{n+1} C^+$ be the $n$ times iterated $\delta$-map. Let

$u = \sum_i x_i \otimes y_i \in C^+ \otimes C^+$

be an element satisfying

(a) $\Sigma_i x_i \otimes y_i = \Sigma_i y_i \otimes x_i$,

(b) $\Sigma_i \delta(x_i) \otimes y_i = \Sigma_i x_i \otimes \delta(y_i)$.

There is a pointed irreducible cocommutative coalgebra $C^u = C \oplus kz$ which contains $C$ as a subcoalgebra and satisfies

$\Delta(z) = z \otimes 1 + 1 \otimes z + u, \quad e(z) = 0$.

Then $V(z)$ is determined by $u$ as follows: $\Sigma_i \delta_{p-2}(x_i) \otimes y_i$ is a symmetric tensor in $\otimes^p C^+$. Let

$v: (\text{the symmetric tensors in } \otimes^p C^+) \to k^{-1/p} \otimes C^+$

be the $1/p$-linear map defined [1, Theorem 4.1.1(a), p. 273] (where denoted by $V$). Put $v(u) = v(\Sigma_i \delta_{p-2}(x_i) \otimes y_i)$. Then $V(z)$ is precisely $v(u)$.

If $C$ underlies a Hopf algebra, then the image $\delta(C^+)$ can be characterized as follows.

**Theorem C.** Let $H$ be an irreducible cocommutative Hopf algebra. The image $\delta(H^+)$ is precisely the set of elements $u$ in $H^+ \otimes H^+$ satisfying (a), (b) and (c) $v(u) \in V(H)$.

**Proof.** If $u = \delta(x)$ with $x \in H^+$, then $u$ satisfies (a), (b) and $v(u) = V(x) \in V(H)$. Conversely, if $u$ satisfies (a), (b), (c), let $\tilde{H}$ be the Hopf algebra generated by $H$ and one indeterminate $z$ with $\delta(z) = u$, $e(z) = 0$. It follows from $V(z) = v(u) \in V(H)$ that $V(H) = V(\tilde{H})$. Hence $\delta(z) = \delta(x)$ for some $x \in H$ by Theorem B. Q.E.D.
Theorem A follows from Theorem C applied to $u = \sum_{i=1}^{\ell-1} x_i \otimes x_{\ell-i}$ and $V^{\otimes -i}(H)$ as $H$.

REFERENCES


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