TWO APPLICATIONS OF ASYMPTOTIC PRIME DIVISORS

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Abstract. Some recent interest has focused on the set of prime divisors of large powers of an ideal in a Noetherian ring. This note presents two results whose proofs appear to depend on knowledge of such asymptotic prime divisors.

Introduction. Let $I$ be an ideal in a Noetherian ring $R$. It was recently shown that, for all large $n$, $\text{Ass}(R/I^n) = \text{Ass}(R/I^{n+1})$ [1]. Many interesting ideas have ensued. For example, we prove the following two results.

Theorem A. Let $\overline{R}$ be the integral closure of the Noetherian domain $R$. If $J$ is a finitely generated ideal of $\overline{R}$, then only finitely many primes of $\overline{R}$ are minimal over $J$.

Theorem B. Let $R \subseteq T$ be an integral extension of domains with $R$ Noetherian. If $Q$ is prime in $T$ and height $Q = n$, then grade $Q \cap R < n$. Furthermore, if grade $Q \cap R = n$, then $Q \cap R$ is a prime divisor of any ideal generated by a maximal $R$-sequence from $Q \cap R$.

Needing only a fraction of the existing knowledge of asymptotic prime divisors, we present it, rather than just giving references.

Lemma [5]. Let $I$ be an ideal in a Noetherian ring $R$. The set $\bigcup \text{Ass}(R/I^n)$, $n = 1, 2, \ldots$, is finite.

Proof. Let $t$ be an indeterminate and let $A = R[t^{-1}, It]$, the Rees ring. Now $t^{-n}A \cap R = I^n$, and if $P \in \text{Ass}(R/I^n)$ one easily finds $Q \in \text{Ass}(A/t^{-n}A)$ with $Q \cap R = P$. As $t^{-1}$ is regular, $Q \in \text{Ass}(A/t^{-1}A)$, which is a finite set.

Lemma [3]. Let $R \subseteq T$ be an integral extension of domains, $R$ Noetherian. Let $I$ be an ideal of $R$ and let $Q \in \text{Spec} T$ with $Q$ minimal over $IT$. Then $P = Q \cap R \in \bigcup \text{Ass}(R/I^n)$.

Proof. We may assume $R$ is local at $P$. We also assume $T = R[u]$ with $u \in \overline{R}$. To do this, by going up assume $T = \overline{T}$, and then by going down assume $T = \overline{R}$. Finally, choose $u \in Q$ but in no other prime of $\overline{R}$ lying over $P$. Thus only $Q$ lies over $Q \cap R[u]$, and so we assume $T = R[u]$. 

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Pick \(0 \neq b \in R\) with \(bT \subseteq R\), and \(n\) large enough that \(b \in Q^n\). As \(Q\) is minimal over \(I^nT\), there is a \(k > 0\) and an \(s \in T - Q\) with \(sQ^k \subseteq I^nT\). Thus \(bsP^k \subseteq bsQ^k \subseteq bsI^nT \subseteq I^n\), since \(bT \subseteq R\). However \(bs \in R - I^n\), since if \(bs \in I^n \subseteq Q^n\), then since \(Q^n\) is primary to the maximal \(Q\), \(b \in Q^n\) a contradiction. Therefore \(P^k\) consists of zero divisors modulo \(I^n\), and being maximal, \(P \in \text{Ass}(R/I^n)\).

**Proof of Theorem A.** Let \(J = (a_1, \ldots, a_m)\bar{R}\). Let \(R_1 = R[a_1, \ldots, a_m]\) and \(I = (a_1, \ldots, a_m)R_1\). Since \(J = J\bar{R}\), if \(Q \in \text{Spec} \bar{R}\) and \(Q\) is minimal over \(J\), then \(Q \cap R \in \bigcup \text{Ass}(R_1/I^nR_1)\). The first lemma, and the fact that only finitely many primes of \(\bar{R}\) lie over any prime of \(R_1\), give the result.

**Proof of Theorem B.** Induct on \(n\). If \(n = 1\), pick \(0 \neq a \in Q \cap R\). Thus \(Q\) is minimal over \(aT\), so \(Q \cap R \in \text{Ass}(R/a^mR)\), some \(m\). Therefore \(Q \cap R \in \text{Ass}(R/aR)\). For \(n > 1\), suppose grade \(Q \cap R > n - 1\) and let \(a_1, \ldots, a_n\) be an \(R\)-sequence from \(Q \cap R\). By induction, we see that height\((a_1, \ldots, a_n)T > n\). Thus \(Q\) is minimal over \((a_1, \ldots, a_n)T\) so that \(Q \cap R\) is a prime divisor of \((a_1R, \ldots, a_nR)^m\), some \(m\). As \(a_1, \ldots, a_n\) is an \(R\)-sequence, \(Q \cap R\) is also a prime divisor of \((a_1, \ldots, a_n)R\) [2, §3.1, Exercise 13].

Theorem B extends [4, 33.11].

**Added in proof.** A recently discovered sophisticated argument shows that in Theorem B, height \(Q = n\) can be weakened to little height \(Q = n\).

**References**


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