PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 2, February 1982

# TWO APPLICATIONS OF ASYMPTOTIC PRIME DIVISORS

STEPHEN McADAM[1]

ABSTRACT. Some recent interest has focused on the set of prime divisors of large powers of an ideal in a Noetherian ring. This note presents two results whose proofs appear to depend on knowledge of such asymptotic prime divisors.

**Introduction.** Let $I$ be an ideal in a Noetherian ring $R$. It was recently shown that, for all large $n$, $\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I^{n+1})$ [1]. Many interesting ideas have ensued. For example, we prove the following two results.

THEOREM A. *Let $\bar{R}$ be the integral closure of the Noetherian domain $R$. If $J$ is a finitely generated ideal of $\bar{R}$, then only finitely many primes of $\bar{R}$ are minimal over $J$.*

THEOREM B. *Let $R \subseteq T$ be an integral extension of domains with $R$ Noetherian. If $Q$ is prime in $T$ and height $Q = n$, then grade $Q \cap R \leqslant n$. Furthermore, if grade $Q \cap R = n$, then $Q \cap R$ is a prime divisor of any ideal generated by a maximal $R$-sequence from $Q \cap R$.*

Needing only a fraction of the existing knowledge of asymptotic prime divisors, we present it, rather than just giving references.

LEMMA [5]. *Let $I$ be an ideal in a Noetherian ring $R$. The set $\bigcup \operatorname{Ass}(R/I^n)$, $n = 1, 2, \ldots,$ is finite.*

PROOF. Let $t$ be an indeterminate and let $A = R[t^{-1}, It]$, the Rees ring. Now $t^{-n}A \cap R = I^n$, and if $P \in \operatorname{Ass}(R/I^n)$ one easily finds $Q \in \operatorname{Ass}(A/t^{-n}A)$ with $Q \cap R = P$. As $t^{-1}$ is regular, $Q \in \operatorname{Ass}(A/t^{-1}A)$, which is a finite set.

LEMMA [3]. *Let $R \subset T$ be an integral extension of domains, $R$ Noetherian. Let $I$ be an ideal of $R$ and let $Q \in \operatorname{Spec} T$ with $Q$ minimal over $IT$. Then $P = Q \cap R \in \bigcup \operatorname{Ass}(R/I^n)$.*

PROOF. We may assume $R$ is local at $P$. We also assume $T = R[u]$ with $u \in \bar{R}$. To do this, by going up assume $T = \bar{T}$, and then by going down assume $T = \bar{R}$. Finally, choose $u \in Q$ but in no other prime of $\bar{R}$ lying over $P$. Thus only $Q$ lies over $Q \cap R[u]$, and so we assume $T = R[u]$.

Received by the editors March 16, 1981 and, in revised form, May 19, 1981.
1980 *Mathematics Subject Classification*. Primary 13E05.
*Key words and phrases*. Noetherian domain, prime divisor, asymptotic prime divisor, integral extension.
[1]Partially supported by the National Science Foundation.

Pick $0 \neq b \in R$ with $bT \subseteq R$, and $n$ large enough that $b \notin Q^n$. As $Q$ is minimal over $I^nT$, there is a $k > 0$ and an $s \in T - Q$ with $sQ^k \subseteq I^nT$. Thus $bsP^k \subseteq bsQ^k \subseteq bsI^nT \subseteq I^n$, since $bT \subseteq R$. However $bs \in R - I^n$, since if $bs \in I^n \subseteq Q^n$, then since $Q^n$ is primary to the maximal $Q$, $b \in Q^n$ a contradiction. Therefore $P^k$ consists of zero divisors modulo $I^n$, and being maximal, $P \in \operatorname{Ass}(R/I^n)$.

PROOF OF THEOREM A. Let $J = (a_1, \ldots, a_m)\bar{R}$. Let $R_1 = R[a_1, \ldots, a_m]$ and $I = (a_1, \ldots, a_m)R_1$. Since $J = I\bar{R}$, if $Q \in \operatorname{Spec} \bar{R}$ and $Q$ is minimal over $J$, then $Q \cap R \in \bigcup \operatorname{Ass}(R_1/I^nR_1)$. The first lemma, and the fact that only finitely many primes of $\bar{R}$ lie over any prime of $R_1$, give the result.

PROOF OF THEOREM B. Induct on $n$. If $n = 1$, pick $0 \neq a \in Q \cap R$. Thus $Q$ is minimal over $aT$, so $Q \cap R \in \operatorname{Ass}(R/a^mR)$, some $m$. Therefore $Q \cap R \in \operatorname{Ass}(R/aR)$. For $n > 1$, suppose grade $Q \cap R > n - 1$ and let $a_1, \ldots, a_n$ be an $R$-sequence from $Q \cap R$. By induction, we see that height$(a_1, \ldots, a_n)T \geqslant n$. Thus $Q$ is minimal over $(a_1, \ldots, a_n)T$ so that $Q \cap R$ is a prime divisor of $(a_1R, \ldots, a_nR)^m$, some $m$. As $a_1, \ldots, a_n$ is an $R$-sequence, $Q \cap R$ is also a prime divisor of $(a_1, \ldots, a_n)R$ [**2**, §3.1, Exercise 13].

Theorem B extends [**4**, 33.11].

ADDED IN PROOF. A recently discovered sophisticated argument shows that in Theorem B, height $Q = n$ can be weakened to little height $Q = n$.

## REFERENCES

1. M. Brodmann, *Asymptotic stability of* $\operatorname{Ass}(R/I^n)$, Proc. Amer. Math. Soc. (to appear).
2. I. Kaplansky, *Commutative rings*, Univ. of Chicago Press, Chicago, Ill., 1974.
3. S. McAdam, *Asymptotic prime divisors and going down*, Pacific J. Math. **91** (1980), 179–186.
4. M. Nagata, *Local rings*, Interscience, New York, 1962.
5. L. J. Ratliff, Jr., *On prime divisors of* $I^n$, *n large*, Michigan Math. J. **23** (1976), 337–352.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712