PSEUDOHOLOMORPHIC FUNCTIONS WITH NONANTITHOLOMORPHIC CHARACTERISTICS

AKIRA KOOHARA

Abstract. Let \( \kappa(z) \in C^\infty(\Omega) \) and \( \| \kappa \| < 1 \). Necessary and sufficient conditions for the system of equations \( \delta f = \kappa(z) \delta f \) to be locally plentiful are given, and under them a representation of \( \kappa \) also is given.

1. Introduction. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and let \( C^\infty(\Omega) \) denote the space of infinitely differentiable complex valued functions on \( \Omega \). Let \( a \) and \( b \) be in \( C^\infty_{(1,0)}(\Omega) \), the space of \( C^\infty \) differential forms of type \((1, 0)\) on \( \Omega \). Now, consider the \( \mathbb{R} \)-linear mapping \( \nu : C^\infty(\Omega) \rightarrow C^\infty_{(1,0)}(\Omega) \) defined by \( \nu(f) = \bar{\partial} f - \bar{\partial} \bar{a} - j_b \) for \( f \in C^\infty(\Omega) \), where \( \bar{\partial} \) denotes the operator \( \sum_{j=1}^n \partial j_j \) with \( \partial j = \partial / \partial z_j \). Then, \( \text{Ker} \, \nu \), the kernel of the map \( \nu \), is an \( \mathbb{R} \)-submodule of \( C^\infty(\Omega) \).

Quite recently there has been increasing interest in \( \text{Ker} \, \nu \), whose elements are called generalized analytic functions of several complex variables (see [5, 6, 7, 8] and references cited in [7]). We call the equation \( \nu(f) = 0 \) the generalized Cauchy-Riemann equation.

Magomedov and Paramodov [6] introduced the idea of the plentifulness of \( \text{Ker} \, \nu \) to obtain the integrability conditions of the equation \( \nu(f) = 0 \) with \( a = 0 \) on \( \Omega \). When \( \dim \mathbb{R} \text{Ker} \, \nu \) is infinite on \( \Omega \), \( \text{Ker} \, \nu \) is said to be plentiful on \( \Omega \). The plentifulness on \( \Omega \) leads to a complex foliation of codimension one of \( \Omega \) determined by the form \( b \). The null sets of generalized analytic functions are leaves of this foliation.

In [3] the author treated generalized analytic functions under the conditions on \( b \) such that a complex foliation of codimension one of \( \Omega \) follows from them.

In this paper we are concerned with the \( \mathbb{R} \)-linear mapping \( \alpha : C^\infty(\Omega) \rightarrow C^\infty_{(1,0)}(\Omega) \) defined by

\[
\alpha(f) = \sum_{j=1}^n \left\{ \kappa(z) \partial_j f - \partial_j \bar{f} \right\} \, dz_j, \quad \text{for} \, f, \kappa \in C^\infty(\Omega).
\]

\( \text{Ker} \, \alpha \) also is an \( \mathbb{R} \)-submodule of \( C^\infty(\Omega) \) as the map \( \nu \). The equation \( \alpha(f) = 0 \) was investigated by S. Hitotumatu [2] and by the author [4]. The former used function-theoretic methods and the latter differential equation-theoretical ones.

In [4], given some conditions upon the coefficient \( \kappa \), we discussed properties of elements of \( \text{Ker} \, \alpha \) (which we call pseudoholomorphic functions with characteristic
k) similar to those of holomorphic functions and obtained a local representation theorem of such functions.

Now, by using the results in [6, 7] we can obtain necessary and sufficient conditions for Ker \( \alpha \) to be plentiful, because the system of elliptic differential equations \( \alpha(f) = 0 \) can be reduced to the system of type \( \nu(g) = 0 \) by \( f = g + \bar{g} \). However, the methods and results of [6, 7] are not effectual ones to clarify completely structures of Ker \( \alpha \).

The purpose of this paper is to investigate relations between \( \dim_k \text{Ker} \alpha \) and coefficient \( k \), and to give a local representation of \( k \) in the case where Ker \( \alpha \) is plentiful.

2. Preliminaries and notations. Since Ker \( \alpha \) is an \( R \)-submodule of \( C^\infty(\Omega) \), following Magomedov and Paramodov, when \( \dim_k \text{Ker} \alpha \) is infinite, we say that Ker \( \alpha \) or the system of differential equations

\[
\partial_j \tilde{f} = \kappa(z) \partial_j f, \quad j = 1, 2, \ldots, n,
\]

is plentiful on \( \Omega \).

To attain our objective we need a few assumptions on characteristic \( \kappa \). First we assume \( ||\kappa|| = \sup_{z \in \Omega} |\kappa(z)| < 1 \).

If \( \partial \kappa \) vanishes on an open subset \( U \) in \( \Omega \), then, considering the restriction of \( \alpha \) to \( U \) denoted by \( \alpha\mid_U \), we can see that Ker \( (\alpha\mid_U) \) is plentiful [4]. Or, if \( \kappa \) vanishes on \( U \), then (2.1) is the Cauchy-Riemann equations on \( U \). By these reasons we may assume that, for nowhere dense subsets \( E_1 \) and \( E_2 \) of \( \Omega \),

\[
\kappa \neq 0 \quad \text{on } \Omega \backslash E_1, \quad \partial \kappa \neq 0 \quad \text{on } \Omega \backslash E_2.
\]

Let \( C^\infty_{(p,q)}(\Omega) \) denote the space of \( C^\infty \) differential forms of type \( (p, q) \) on \( \Omega \).

We shall define the \( R \)-linear mapping \( \alpha^* \) of \( C^\infty(\Omega) \) into \( C^\infty_{(0,1)}(\Omega) \) by

\[
\alpha^*(f) = \sum_{j=1}^n \left\{ \kappa(z) \partial_j \tilde{f} - \bar{\partial}_j \tilde{f} \right\} d\bar{z}_j = \alpha(f).
\]

Then, we may regard \( \alpha \) and \( \alpha^* \) as \( R \)-linear differential operators of first order on \( C^\infty(\Omega) \).

Let \( \sigma \) be a vector field on \( U \) and \( f \) in \( C^\infty(U) \). When \( \sigma f = 0 \) and \( \sigma \tilde{f} = 0 \) on \( U \), we say that the vector field \( \sigma \) is tangential to \( f \). And when, for every \( f \in \text{Ker}(\alpha\mid_U) \), \( \sigma \) is tangential to \( f \), we say that \( \sigma \) is tangential to \( \text{Ker}(\alpha\mid_U) \).

To seek vector fields tangential to \( \text{Ker}(\alpha\mid_U) \), we need to construct three \( C \)-linear mappings \( \beta, \bar{\beta} \) and \( \theta: C^\infty(\Omega) \rightarrow C^\infty_{(p,q)}(\Omega) \) such that their kernels contain Ker \( \alpha \) and Ker \( \alpha^* \).

Rewriting the map \( \alpha \) by using \( \partial \), we have \( \alpha(f) = \kappa(z) \partial f - \bar{\partial} \tilde{f} \) for \( f \in C^\infty(\Omega) \).

Then we have readily the \( C \)-linear mapping \( \partial \alpha: C^\infty(\Omega) \rightarrow C^\infty_{(2,0)}(\Omega) \) defined by

\[
\partial \alpha(f) = \partial \kappa \wedge \partial f \quad \text{for } f \in C^\infty(\Omega).
\]

We put

\[
\beta = \partial \alpha \quad (= \partial \kappa \wedge \partial).
\]

Then we obtain

\[
\beta(f) = \kappa \partial \alpha(f) - \partial \kappa \wedge \alpha(f).
\]
We thus define the mapping $\bar{\beta} : C^\infty(\Omega) \to C^\infty_{0,2}(\Omega)$ as

\begin{equation}
(2.6) \quad \bar{\beta}(f) = \bar{\beta}(\bar{f}) \quad \text{for } f \in C^\infty(\Omega).
\end{equation}

From the definition of $\alpha^*$ and (2.3)--(2.6) we obtain

\begin{equation}
(2.7) \quad \text{Ker } \alpha^* = \text{Ker } \alpha, \quad \text{Ker } \alpha \subset \text{Ker } \beta = \text{Ker } \bar{\beta}.
\end{equation}

Lastly, we want to construct a mapping $\theta$ of $C^\infty(\Omega)$ into $C^\infty_{0,2}(\Omega)$. To do this, we need the identity: for $f \in C^\infty(\Omega)$

\begin{equation}
\kappa \{ \bar{\partial} \partial \alpha(f) + \partial \kappa \wedge \partial \alpha^*(f) - \kappa \bar{\partial} \beta(f) - \kappa \bar{\partial} \partial \kappa \wedge \alpha(f) \} - \partial \kappa \wedge \bar{\partial} \kappa \wedge \alpha^*(f)
\end{equation}

\begin{equation}
= \kappa (1 - |\kappa|^2) \bar{\partial} \partial \kappa \wedge \partial f + \partial \kappa \wedge \bar{\partial} \kappa \wedge \bar{\partial} f.
\end{equation}

Then, $\theta$ is defined as follows:

\begin{equation}
(2.8) \quad \theta(f) = \kappa (1 - |\kappa|^2) \bar{\partial} \partial \kappa \wedge \partial f + \partial \kappa \wedge \bar{\partial} \kappa \wedge \bar{\partial} f \quad \text{for } f \in C^\infty(\Omega),
\end{equation}

where $\bar{\partial} = \sum_{j=1}^n \partial_j \bar{z}_j$, $\partial_j = \partial / \partial z_j$.

The three mappings defined above may be regarded as $C$-linear differential operators of first order on $C^\infty(\Omega)$.

It follows from (2.7) and the above identity that

\begin{equation}
(2.9) \quad \text{Ker } \alpha \subset \text{Ker } \theta.
\end{equation}

For the purposes of later convenience, we now express (2.4) and (2.8) in terms of coordinates in $C^n$.

We put

\begin{equation}
\kappa_i = \partial_i \kappa, \quad \beta_{ij} = \kappa_i \partial_j - \kappa_j \partial_i,
\end{equation}

\begin{equation}
\gamma_{ijk} = (\partial_k \kappa_i) \partial_j - (\partial_k \kappa_j) \partial_i,
\end{equation}

\begin{equation}
\theta_{ijk} = \kappa (1 - |\kappa|^2) \gamma_{ijk} + \beta_{ij}(\kappa) \partial_k.
\end{equation}

From now on, the indices $i, j$ and $k$ (with or without subscripts) run over the set \{1, 2, \ldots, n\} unless specifically stated otherwise.

Then we have

\begin{equation}
\beta = \partial \kappa \wedge \partial = \sum_{i<j} \{ (\partial_i \kappa) \partial_j - (\partial_j \kappa) \partial_i \} \ dz_i \wedge \ dz_j,
\end{equation}

\begin{equation}
\theta = \sum_{i<j, k} \theta_{ijk} \ dz_i \wedge \ dz_j \wedge \ dz_k.
\end{equation}

In the following section we shall prove that on $\Omega$

\begin{equation}
(2.10) \quad \bar{\partial} \partial \kappa \wedge \partial \kappa = 0, \quad \partial \kappa \wedge \partial \kappa = 0.
\end{equation}

In terms of coordinates of $C^n$ we rewrite the left sides of (2.10).

\begin{equation}
\bar{\partial} \partial \kappa \wedge \partial \kappa = \sum_{i<j, k} \gamma_{ijk}(\kappa) \ dz_i \wedge \ dz_j \wedge \ dz_k,
\end{equation}

\begin{equation}
\partial \kappa \wedge \partial \kappa = \sum_{i<j} \beta_{ij}(\kappa) \ dz_i \wedge \ dz_j.
\end{equation}
3. Necessary conditions for plentifulness. Let \( w \) be a nonconstant pseudoholomorphic function on \( \Omega \). By the unique continuation property for pseudoholomorphic functions [4], we have a nowhere dense subset \( E_3 \) of \( \Omega \) such that \( \partial w \neq 0 \) on \( \Omega \setminus E_3 \). If we put \( E = E_1 \cup E_2 \cup E_3 \), \( \kappa \neq 0 \), \( \partial \kappa \neq 0 \) and \( \partial w \neq 0 \) on \( \Omega \setminus E \). Since \( \partial \kappa \wedge \partial w = 0 \) on \( \Omega \), for any \( z \in \Omega \setminus E \) there is a number \( i' \) such that \( \partial_{i'} \kappa \neq 0 \) and \( \partial_{i'} w \neq 0 \).

Though we must prove (2.10) about each point of \( \Omega \setminus E \), it is enough to prove it about a specific point. We may assume without loss of generality that if \( 0 \in \Omega \setminus E \), then \( w(0) = 0 \), \( \partial_{i'} \kappa(0) \neq 0 \) and \( \partial_{i'} w(0) \neq 0 \).

To prove (2.10) we use the following special change of variables on a small neighborhood \( U \) of the origin

\[
(3.1) \quad \xi_j = z_j, \quad j = 1, 2, \ldots, n - 1, \quad \xi = w(z).
\]

This is nonsingular because \( w \) satisfies (2.1).

We put \( \partial_j' = \partial / \partial \xi_j, \quad \partial_j'' = \partial / \partial \xi_j \), \( \Delta = \kappa(1 - |\kappa|^2) \) and \( \kappa_k = \partial_k \kappa \). Moreover we denote by \( [ \quad ]' \) the functions into which ones in \( [ \quad ] \) are transformed by (3.1). If we note (2.9), i.e. \( \theta_{ijk}(f) = 0 \) for any \( f \in \text{Ker } \alpha \), \( \theta_{ijk} \) are transformed into the following on \( U \):

\[
\begin{align*}
[\theta_{ijk}]'' &= \Delta''\left\{ [\kappa_{ik}]''\partial_j' - [\kappa_{jk}]''\partial_i' \right\} + [\beta_{ij}(\kappa)]''\partial_k + [\theta_{ijk}(w)]''\partial_{i'} \\
&\quad \text{for } i \neq n, j \neq n, k \neq n, \\
[\theta_{njk}]'' &= [\Delta \kappa_{nk}]''\partial_j' + [\beta_{nj}(\kappa)]''\partial_k + [\theta_{njk}(w)]''\partial_{i'} \\
&\quad \text{for } j \neq n, k \neq n, \\
[\theta_{ink}]'' &= -[\Delta \kappa_{nk}]''\partial_j' + [\beta_{in}(\kappa)]''\partial_k + [\theta_{ink}(w)]''\partial_{i'} \\
&\quad \text{for } i \neq n, k \neq n, \\
[\theta_{ijn}]'' &= \Delta''\left\{ [\kappa_{in}]''\partial_j' - [\kappa_{jn}]''\partial_i' \right\} + [\theta_{ijn}(w)]''\partial_{i'} \\
&\quad \text{for } i \neq n, j \neq n.
\end{align*}
\]

**Lemma 1.** If \( \text{Ker } \alpha \) has an element \( W \) linearly independent of \( w \), then the vector fields \( \theta_{ijk} \) are tangential to \( w \).

**Proof.** It is sufficient to show \( \theta_{ijk}(\bar{w}) = 0 \). Assume there are a point \( z' \) and numbers \( i', j', k' \) such that \( \theta_{ijk}(\bar{w}) \neq 0 \) at \( z' \). We may regard \( z' \) as the origin and, shrinking \( U \) mentioned above if necessary, assume that \( \theta_{ijk}(\bar{w}) \neq 0 \) on \( U \). Using the coordinates introduced in (3.1), we obtain, on the image of \( U \) by (3.1),

\[
\begin{align*}
\partial_j' W'' &= 0, \quad \partial_j'' W'' = 0 \quad (j = 1, \ldots, n - 1) \quad \text{and} \quad \partial_i'' W'' = 0,
\end{align*}
\]

where we use the relation derived from (3.2), \( \theta_{ijk}(W) = [\theta_{ijk}(w)]''\partial_i'' W'' \). Therefore we see \( W'' \) depends only on \( \xi \) and is holomorphic at 0.

However, since \( \alpha(W) = 0 \), \( [\kappa_i w]''(\partial_i W'' = \partial_i \bar{W}'') = 0 \), and hence \( \partial_i W'' = \partial_i \bar{W}''). \) Thus we obtain \( \bar{W} = aw + b \) on \( U \), where \( a > 0 \) is a constant and \( b \in C \), which contradicts the assumption.

Let \( S \) be any subset of \( \Omega \). The set \( N(S) \) of those vector fields on \( S \) which are tangential to \( w \) is a \( C^\infty(S) \)-submodule of the \( C^\infty(S) \)-module \( M(S) \) consisting of all vector fields on \( S \).

If \( f \in \text{Ker } \alpha \) is nonconstant, nonempty level sets \( \{ z \in \Omega | f(z) = \text{const.} \} \) are \( (n - 1) \)-dimensional complex submanifolds except the set of nonordinary points of \( f \) [2].
By virtue of (2.7) every vector field $\beta_j$ is tangential to $\text{Ker} \alpha$. If $0 \in \Omega \setminus E$, letting $\beta_i$ denote $\beta_{in}$ ($i = 1, \ldots, n - 1$), we see from the above-mentioned that $\{ \beta_i, \bar{\beta}_i \}$ span $N(U)$.

We say $\text{Ker} \alpha$ is trivial when it is $C$ itself.

**Lemma 2.** Under the same assumption as in Lemma 1, (2.10) holds on $\Omega$.

**Proof.** Lemma 1 shows $\theta_{ijk} \in N(\Omega)$. Assume $0 \in \Omega \setminus E$. Then $\theta_{ijk}$ can be written by linear combinations of $\beta_s, \bar{\beta}_s$ ($s = 1, \ldots, n - 1$) with coefficients in $C^\infty(U)$. If $a^i_{jk}, b^i_{jk}$ denote the coefficients of $\beta_s, \bar{\beta}_s$, we have:

For $1 < i < j < n - 1$ and each $k$,

$$a^i_{jk} = 0, \quad 1 < s < n - 1, s \neq i, j,$$

$$\kappa_n a^i_{jk} = \Delta \kappa_{jk}, \quad \kappa_n a^i_{jk} = -\Delta \kappa_{ik}, \quad \sum_{s=1}^{n-1} a^i_{jk} \kappa_s = 0. \tag{3.3}$$

For $1 < i < j = n$ and each $k$,

$$a^i_{nk} = 0, \quad 1 < s < n - 1, s \neq i,$$

$$\kappa_n a^i_{nk} = \Delta \kappa_{nk}, \quad \kappa_n a^i_{nk} = \Delta \kappa_{ik}. \tag{3.4}$$

For $1 < k < n - 1$ and each $i, j$,

$$b^i_{jk} = 0, \quad 1 < s' < n - 1, s' \neq k,$$

$$\kappa_k b^k_{ijk} = 0, \quad \kappa_k b^k_{ijk} = -\beta_{ij}(\bar{k}). \tag{3.5}$$

For $k = n$ and each $i, j, \beta_{ij}(\bar{k}) = 0$ by $\beta_{jn} = 0, 1 < s' < n - 1$.

If $k \neq n$, from (3.5) we need to consider the following two cases.

Case 1. For all $k, 1 < k < n - 1, \kappa_k = 0$ on $U$.

Case 2. For some $k', 1 < k' < n - 1$ and some point $z' \in U, \kappa_{k'} = 0$ at $z'$.

We prove the first part of (2.10) only in Case 1. Let there be an open subset $V$ of $U$ and some $i', j'$ ($i' < j'$) such that $\beta_{ij'}(\bar{k'}) = 0$. Then, $\beta_i = \partial_i, i = 1, \ldots, n - 1$, and so

$$\beta_i = \left[ \beta_{ij'}(\bar{k'}) \right]^{-1} \left( \theta_{ij'k} + (\Delta \kappa_{ij'} \beta_{ij'}) \right) \in N(V).$$

Hence $N(V) = M(V)$, which contradicts the nontrivial Ker $\alpha$.

We next prove the second part of (2.10). From (3.3) and (3.4) we have $\gamma_{ijk} = \kappa_{ik} \kappa_j - \kappa_{jk} \kappa_i = 0$ on $U$, which completes the proof.

**Corollary.** Let $U$ be an open subset of $\Omega$. If on $U$ either (1) $\partial k \wedge \partial k \neq 0$, $\bar{\partial} \partial k \wedge \partial k = 0$ or (2) $\partial k \wedge \partial k = 0, \bar{\partial} \partial k \wedge \partial k \neq 0$, then Ker $\alpha$ is trivial.

**Proof.** Let $w \in \text{Ker} \alpha$ be nonconstant.

Case (1). $\bar{\partial} \partial k \wedge \partial k = 0$ leads to $\theta(w) = \partial k \wedge \partial k \wedge \bar{\partial} w$. By (2.7) and (2.9), $\bar{\partial} w = 0$ on $U, w$ is constant.

Case (2). $\partial k \wedge \partial k = 0$ leads to $\theta(w) = \bar{\partial} \partial k \wedge \partial k$. Since $\partial k \wedge \partial w = 0, \theta(w) = c \bar{\partial} \partial k \wedge \partial k$ for some function $c \in C^\infty(U)$, and hence $\bar{\partial} \partial k \wedge \partial k = 0$ on $U$, which contradicts the assumption.

**Theorem 1.** For the system (2.1) to be plentiful on $\Omega$, it is necessary that the characteristic $\kappa$ fulfill the condition (2.10) on $\Omega$. 

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4. Sufficient conditions for plentifulness. We show the local validity of the
converse of Theorem 1. As is readily verified, the first half of (2.10) is sufficient for
the $(1, 0)$-form $\sum \kappa_j dz_j$ to determine a complex foliation of codimension one of $\Omega$. The converse of this is not always valid (see, e.g. Example (ii) below).

**Lemma 3.** For a function $\kappa \in C^\infty(\Omega)$ satisfying $\partial \kappa \neq 0$ and (2.10) on $\Omega$, there is locally a holomorphic function $h$ such that $dh \wedge \partial \kappa = 0$, $dh \wedge \partial \kappa = 0$ and $dh \neq 0$.

**Proof.** If we put $\omega = \partial \kappa$, by the first half of (2.10) $\tilde{\omega} = \rho \wedge \omega$ for a form $\rho \in C^\infty(0,1)(\Omega)$. From this $\tilde{\rho} \wedge \omega = 0$, $\omega \neq 0$ leads to $\tilde{\rho} = 0$ on $\Omega$, so that for each point of $\Omega$ there are a neighborhood $U$ of that point and a function $g \in C^\infty(U)$ such that $\tilde{\rho} = \rho$. Putting $\tau = \omega \exp(-g)$, we see $\tilde{\tau} = 0$, which shows $\tau$ is a holomorphic form. By using $\omega = \tau \exp g$, we have $\partial g \wedge \tau + d\tau = 0$, and hence

\begin{equation}
\tau \wedge d\tau = 0, \quad \tau \neq 0 \quad \text{on } U.
\end{equation}

Let $H(U)$ denote the algebra of all holomorphic functions on $U$. We define $\tau = \sum \tau_j dz_j$ and $D_{ij} = \tau_i \partial_j - \tau_j \partial_i$, $\tau_j \in H(U)$. Then, by (4.1) we have a function $h$ holomorphic on a neighborhood $V \subset U$ such that $dh \neq 0$ and $\tau \wedge dh = 0$ (i.e. $D_{ij}$ is tangential to $h$). Thus the proof is complete.

**Lemma 4 [4, Theorem 20].** Assume that $\kappa$ satisfies (2.10) and $\partial \kappa \neq 0$ on $\Omega$. Then (2.1) is locally reduced to the equation of one variable

\begin{equation}
\partial_z F = K(t) \partial_t F, \quad |K| < 1,
\end{equation}

where $K(t)$ is defined and of class $C^\infty(h(V))$.

**Proof.** We have a holomorphic function $h$ satisfying the conditions of Lemma 3 on an open subset $V$ of $\Omega$. We may assume $h$ is the coordinate function $z_n$. Since $\partial \kappa \wedge dz_n = 0$, $\partial \kappa \wedge dz_n = 0$, $\kappa$ and $\bar{\kappa}$ are holomorphic in the other coordinates when fixing $z_n$, and so is a function of $z_n$ alone. Then $\kappa = K(z_n)$ and, for any $f \in \text{Ker } \alpha$, $\partial f \wedge dz_n = \partial \bar{f} \wedge dz_n = 0$, so $f = F(z_n)$. Thus equation (4.2) is obtained.

We now take a disk $\delta \subset h(V)$. Then we have

**Lemma 5.** Equation (4.2) is plentiful on $\delta$.

**Proof.** Let $\Delta_i$ ($i = 1, 2$) be disks concentric with $\delta$ such that $\delta \subset \Delta_1 \subset \Delta_2$. We take a function $K_1(t) \in C^\infty$ on $C$ as follows: $K_1(t)$ equals one on $\delta$ and zero outside $\Delta_1$. Besides, it fulfills $0 < K_1(t) < 1$. Putting $L(t) = K_1(t)K(t)$, we have the equation

\begin{equation}
\partial_z G = L(t) \overline{\partial_z G}.
\end{equation}

Consider the Dirichlet problem of (4.3) with the boundary conditions $\text{Re } G(t) = g(t)$ on $\partial \Delta_2$, and $G(t_0) = 0$, $t_0 \in \Delta_2$, where $g(t)$ is a given real valued continuous function on $\Delta_2$. This problem is solvable [1]. The plentifulness on $\delta$ of (4.2) is derived from the fact that $\dim_R C(\partial \Delta_2)$ is infinite and from the unique continuation property for solutions of equation (4.3). Thus we have the following

**Theorem 2.** If the characteristic $\kappa$ satisfies (2.10) on $\Omega$, then Ker $\alpha$ is locally plentiful.

Using Lemma 3 and the proof of Lemma 4, we have a local representation of $\kappa$. 

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Theorem 3. There are locally a holomorphic function $h$ and $K(t) \in C^\infty(\text{img } h)$, $t = h(z)$, such that $\kappa = K \circ h$, $dh \neq 0$ and $K \neq 0$ if and only if $\kappa$ satisfies (2.10) and $\partial \kappa \neq 0$ on $\Omega$.

Examples. (i) Consider $\kappa = \phi(z) + \overline{\psi(z)}$, where $\phi$ and $\psi$ are holomorphic and $d\phi \wedge d\psi = 0$ on $\Omega$. Then $\partial \kappa \wedge \overline{\partial \kappa} \neq 0$, even though $\overline{\partial \partial \kappa} \wedge \partial \kappa = 0$, so $\ker \alpha$ is trivial by the corollary to Lemma 2.

(ii) We consider (2.1) on a small neighborhood $U$ of the origin in $\mathbb{C}^2$ such that $w(z)$ is well defined on $U$ by the following equation:

$$w(z) = (\bar{z}_1 + z_2)^2 - 2z_2, \quad w(0) = 0.$$  

Putting $\kappa = w(z) + \bar{z}_1 + z_2$ (restricting $U$ further if necessary) we can observe that $\ker \alpha$ is generated only by $w(z)$ (see also [4]). A simple computation shows that $\partial_1 w - \partial_2 w \neq 0$ on $U$, and so it is easy to obtain $\partial \kappa \wedge \overline{\partial \kappa} = (\partial_1 w - \partial_2 w)(dz_1 \wedge dz_2) \neq 0$. Moreover we can also show

$$\overline{\partial \partial \kappa} \wedge \partial \kappa \neq 0 \quad \text{on } U.$$  

(iii) We regard the function $w$ defined by (4.4) as $\kappa(z)$. Evidently we have $\partial \kappa \wedge \overline{\partial \kappa} = 0$ and (4.5) on $U$. By the corollary to Lemma 2 $\ker \alpha$ is trivial.

(iv) Global plentifulness is not always true, even though $\kappa$ satisfies (2.10) on $\Omega$. The following example shows global plentifulness is valid.

If we take $\kappa = (2/3)(z_1^2 + z_2)$ on $\Omega = \{z \in \mathbb{C}^2 : |z_1^2 + z_2| < 1\}$, then we have easily, for any nonnegative integers $m$,

$$w = (z_1^2 + z_2)^{m+1}/(m+1) + (2/3)(\bar{z}_1^2 + \bar{z}_2)^{m+2}/(m+2).$$

It is trivial for $\ker \alpha$ to be plentiful on $\Omega$.

In conclusion we are in a position to state relations between $\dim_k \ker \alpha$ and $\kappa$.

We define the following notations: $d = \dim_k \ker \alpha$, $\theta_1 = \partial \kappa \wedge \overline{\partial \kappa}$ and $\theta_2 = \overline{\partial \partial \kappa} \wedge \partial \kappa$.

(1) If $d > 3$, $\theta_1 = \theta_2 = 0$ on $\Omega$.

(2) If $\theta_1 = \theta_2 = 0$ on $\Omega$, $d = +\infty$ locally.

(3) If there is an open subset $U$ of $\Omega$ on which either $\theta_1 \neq 0$, $\theta_2 = 0$ or $\theta_1 = 0$, $\theta_2 \neq 0$, then $d = 1$ (the $\ker \alpha$ is trivial).

(4) If there is a point of $\Omega$ at which $\theta_1 \neq 0$, $\theta_2 \neq 0$, then $d < 2$.

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References


Department of Mathematics, Himeji Institute of Technology, Shosha 2167, Himeji, 671-22, Japan