A NONVARIATIONAL SECOND ORDER ELLIPTIC OPERATOR
WITH SINGULAR ELLIPTIC MEASURE

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Abstract. We exhibit an example which proves that the elliptic measure for a
second-order operator of the form $\sum_{i,j=1}^{n} a_{ij} D_{ij}^2 u$ with continuous coefficients can be
singular with respect to the surface measure on the boundary of a smooth
two-dimensional domain.

1. Introduction. In the papers [2] and [4], examples of singular elliptic measures
for second-order operators in divergence form are given. In this note, following the
ideas contained in [4], we exhibit an analogous example in the nonvariational case.

We recall the definition of elliptic measure. Let $\Omega$ be a bounded subset of $\mathbb{R}^n$
with smooth boundary $\partial \Omega$ and $L = \sum_{i,j=1}^{n} a_{ij} D_{ij}^2$, a uniformly elliptic operator with
continuous coefficients in $\overline{\Omega}$. It is well known that, for every $g \in C(\partial \Omega)$, there exists
a unique solution $u \in W^{2,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of the problem
$$Lu = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega.$$ 

The classical maximum principle and the Riesz representation theorem imply
that for each $P \in \Omega$ there exists a Borel measure on $\partial \Omega$, $\omega_L^P$ (the $L$-elliptic measure
evaluated at $P$), such that the following formula holds:
$$u(P) = \int_{\partial \Omega} g(\sigma) \, d\omega_L^P(\sigma).$$

On the other hand, by a result of Pucci and Alexandrov (see [1] and [6]), the
solution, vanishing on the boundary, of the equation $Lv = f, f \in L^n(\Omega)$, satisfies
the following estimate
$$\max_{P \in \Omega} |v(P)| < c \|f\|_{L^n(\Omega)}$$
where the constant $c$ depends only on the ellipticity constants and the geometry of
$\Omega$.

Notice that, by a result of Talenti [7], in dimension two we have the stronger
inequality
$$\|v\|_{W^{2,2}(\Omega)} < c \|f\|_{L^2(\Omega)},$$
where $c$ depends on the same parameters as before.

Pucci-Alexandrov's theorem implies the existence of the Green's function
$G(P; Q)$, such that $G(P; \cdot) \in L^{n/(n-1)}(\Omega)$ for every fixed $P$ and $v(P) = \int_{\Omega} G(P; Q)f(Q) \, dQ.$
In the case of smooth coefficients, the divergence theorem gives the connection between the Green's function and the $L$-elliptic measure; we have

$$d\omega^P_\sigma (\sigma) = \sum_{i,j=1}^n a_{ij}(\sigma) v_i D_j G(P; \sigma) \, d\sigma$$

where $v = (v_1, v_2, \ldots, v_n)$ is the inward unit normal to $\partial \Omega$, and $d\sigma$ is the usual $(n-1)$-dimensional surface measure on $\partial \Omega$.

Furthermore we recall that in $\Omega \setminus \{P\}$, $G(P, \cdot)$ satisfies the equation $L^* G(P; \cdot) = 0$, where $L^*$ is the adjoint operator of $L$; formally $L^* u = \sum_{i,j=1}^n D^2_{ij}(a_{ij} u)$.

2. Main lemmas. In this section, $B$ is a bounded $C^\infty$ domain in the upper half-plane $\mathbb{R}_2^+$ adjacent to the $x$-axis such that

$$B_0 = \{(x, y): |x| < 2, y = 0\} \subseteq \partial B.$$ 

$P$ is a given point in $B$.

**Lemma 1.** Suppose $\beta = \beta(x, y)$, $\beta^h = \beta^h(x, y)$ ($h = 1, 2, \ldots$) are $C^\infty$ functions in $\mathbb{R}^2$ satisfying the following conditions:

(a) $\frac{1}{2} < \beta < \frac{3}{2}$, $\frac{1}{2} < \beta^h < \frac{3}{2}$ \forall $h$;
(b) $|D_y \beta^h(x, y)| < c_1$, $|D_{yy}^2 \beta^h(x, y)| < c_2 \forall (x, y) \in B, \forall h$;
(c) $\beta^h$ converges weakly in $L^2(B)$ to $\beta$, as $h \to \infty$;
(d) $\beta^h(x, y) = \beta^h(x, -y)$.

Denote by $E$ and $E_h$ respectively the operators $D_{xx} + \beta D_{yy}$ and $D_{xx}^2 + \beta D_{yy}^2$.

Claim. If $G(x, y) = G(P; x, y)$ and $G^h(x, y) = G^h(P; x, y)$ are the Green’s functions for $E$ and $E_h$ in $B$ with pole $P$, then

$$D_y G^h(x, 0) \rightarrow D_y G(x, 0) \text{ uniformly for } x \in [-1, 1].$$

**Proof.** Denote by $T_+$ the set $\{(x, y): |x| < 1 + \varepsilon; 0 < y < \mu\}$ with $\varepsilon, \mu$ small enough to ensure $\overline{T_+} \subseteq \overline{B}$, $P \notin \overline{T_+}$. $T_-$ will be the set $\{(x, y): |x| < 1 + \varepsilon, -\mu < y < 0\}$ and $T = T_+ \cup T_-$. In $T_+$ we have

$$\tag{2.1} E^*_h G^h = D_{xx}^2 G^h + D_{yy}^2 (\beta D_y G^h) = 0.$$ 

Furthermore $G^h(x, 0) = 0$ if $|x| < 2$.

The equation (2.1) can be written in the following divergence form:

$$\tag{2.2} D_{xx}^2 G^h + D_y (\beta D_y G^h) = -D_y (D_y \beta D_y G^h).$$

Extend now $G^h(P; x, y)$ to an odd function with respect to $y$ across $y = 0$ and call $\overline{G^h}(P; x, y)$ the extended function; then $\overline{G^h}(P; x, y)$ satisfies in $T$ the equation (2.2). On the other hand, by (1.1), it is easy to show that $\overline{G^h}(P; x, y)$ is equibounded in $L^2(T)$ and, by the hypothesis (b) we have $\|D_y \beta^h\|_{L^2(T)} < c_1$.

Well-known results on divergence form equations imply that $\|\overline{G^h}(P; \cdot)\|_{W^{2,1}(T)} < c$ (independent of $h$) and thus, by Sobolev’s immersion theorem, $\overline{G^h}(P; \cdot)$ is equibounded in $L^p(T)$ for every $p > 2$.

Therefore, Meyers’ theorem (see [3]) implies the existence of $\delta > 0$ such that

$$\|\overline{G^h}(P; \cdot)\|_{W^{1,2,\delta}(T)} < \text{ const} \text{ (independent of } h\).$$
In particular we have
\[(2.3) \quad \|G^h(P; \cdot)\|_{W^{k+\theta}(T_+)} < \text{const} \quad \text{(independent of } h)\].

Differentiating now (2.1) with respect to \(y\) and putting \(v^h = D_y G^h\), we see that \(v^h\) satisfies in \(T_+\) the following divergence form equation:
\[D_{xx} v^h + D_y (\beta^h D_y v^h) = -2D_y(D_y \beta^h v^h) - D_y(D_{yy} \beta^h G^h)\].

By (2.3) and hypothesis (b), the right-hand side is the divergence of an equibounded (in \(L^{2+\theta}(T_+)\)) vector field. Using once more Meyers’ result we deduce that
\[(2.4) \quad u^h \in W^{\theta}(T_+) \quad \text{such that } \|u^h\|_{W^{\theta}(T_+)} < \text{const} \quad \text{(independent of } h)\].

From (2.1) and (2.4) it follows that \(\|G^h(P; \cdot)\|_{W^{k+\theta}(T_+)} < \text{const} \quad \text{(independent of } h)\). Sobolev’s immersion theorem implies now that \(G^h\) actually a subsequence) converges in \(C^1(\overline{T_+})\) to some continuous function \(g(x, y)\). The conclusion of Lemma 1 will follow from the following lemma.

**Lemma 2.** Under the hypothesis of Lemma 1, \(G^h(P; x, y)\) converges weakly in \(L^2(B)\) to \(G(P; x, y)\).

**Proof.** Consider the function
\[(2.5) \quad u^h(P) = \int_B G^h(P; x, y)f(x, y) \, dx \, dy\]
where \(f \in L^2(B)\).

The function \(u^h(P)\) is the solution, vanishing on \(\partial B\), of \(E u = f\) in \(B\). From the result of Talenti [7], we have \(\|u^h\|_{W^{2,\theta}(B)} \leq c\|f\|_{L^2(B)}\) with \(c\) depending only on the geometry of \(B\). Therefore \(u^h\) admits a subsequence converging weakly in \(W^{2,\theta}(B)\) and strongly in \(W^{1,p}(B)\) for every \(p > 2\) to a function \(u \in W^{2,\theta}(B)\), which vanishes on \(\partial B\). We will show that \(u\) is the solution, vanishing on \(\partial B\), of \(E u = f\).

We have \(D_{xx}^2 u^h + \beta^h D_{yy}^2 u^h = f\), and therefore
\[D_{xx}^2 u^h + D_y (\beta^h D_y u^h) = D_y \beta^h D_y u^h + f\].

It is enough to show that \((D_y \beta^h)(D_y u^h)\) converges weakly in \(L^2(B)\) to \((D_y \beta)(D_y u)\) and this is an easy consequence of the following facts: \(\beta^h \to \beta\) in \(L^2(B)\) weakly, \(\|\beta^h\|_{L^\infty(B)} \leq c\), \(D_y u^h \to D_y u\) in \(L^2(B)\) strongly.

On the other hand \(G^h(P; \cdot)\) is equibounded in \(L^2(B)\) and so has a subsequence which converges weakly in \(L^2(B)\) to some function \(v_p \in L^2(B)\). From the representation formula (2.5), letting \(h\) tend to infinity, we have
\[(2.6) \quad u(P) = \int_B v_p(x, y)f(x, y) \, dx \, dy\].

Since (2.6) holds for any \(f \in L^2(B)\), we conclude \(v_p = G\).

**Lemma 3.** Suppose \(\beta\) and \(\beta^h\) are \(C^\infty\) functions in \(R^2\) satisfying the following hypotheses:
(a) \(\frac{1}{2} < \beta < \frac{3}{2}, \quad \frac{1}{2} < \beta^h < \frac{3}{2} \quad \forall h\);
(b) \(|D_x \beta^h(x, y)| < C_1, \quad |D_{xx}^2 \beta^h(x, y)| < C_2 \quad \forall (x, y) \in B, \forall h\);
(c) $\beta^h$ converges to $\beta$ in $L^2_{\text{loc}}(B)$;
(d) $\beta^h(x,y) = \beta(x,-y)$.

Denote by $E$ and $E_h$ the operators $D_{xx}^2 + \beta \cdot D_{xy}^2$ and $D_{xx}^2 + \beta^h \cdot D_{xy}^2$ and by $G(x,y)$ = $G(P;x,y)$, $G^h(x,y) = G^h(P;x,y)$ their Green's functions in $B$ with pole $P$.

Claim. $D_y(\beta^h G^h)(x,0) \to D_y(\beta G)(x,0)$ uniformly for $x \in [-1, 1]$.

**Proof.** Let $T_+$ be as in the proof of Lemma 1. In $T_+$ we have

\[ D_{xx}^2 G^h + D_{yy}^2 (\beta^h G^h) = 0. \]

This means that the function $v^h = \beta^h G^h$ satisfies the equation $D_{xx}^2 (v^h/\beta^h) + D_{yy}^2 v^h = 0$. Furthermore $v^h(x,y) = 0$ if $|x| < 1 + \varepsilon$ and $y = 0$. Arguing now as in Lemma 1, having interchanged the roles of $x$ and $y$, we deduce that $v^h \to v$ in $C^1(\overline{T}_+)$ where $v$ is some $C^1$ function in $\overline{T}_+$.

The lemma will be proved if we show that $D_y v(x,0) = D_y (\beta G)(x,0)$ for $|x| < 1$. But, if we recall that the $L$-elliptic measure $\omega^p_\Omega$ has density $D_y (\beta G)(x,0)$, this follows from the following general lemma.

**Lemma 4.** Let $\Omega$ be a bounded domain of $R^n$ and $L^h = \sum_{i,j=1}^n a_{ij}^h D_{ij}^2$ $(h = 1, 2, \ldots)$ with $a_{ij} \in C(\overline{\Omega})$. Suppose furthermore that
(a) for every $\xi \in R^n$, $\mu |\xi|^2 < \sum_{i,j=1}^n a_{ij}^h \xi_i \xi_j < M |\xi|^2$ with $\mu$ and $M$ independent of $h$;
(b) $a_{ij}^h \to a_{ij}$ in $L^2_{\text{loc}}(\Omega)$.

Then, if $L$ is the limit operator $\sum_{i,j=1}^n a_{ij} D_{ij}^2$ and $P$ is fixed in $\Omega$, $\omega^p_\Omega$ converges weakly to $\omega^p_\Omega$, that is, for every $g \in C(\partial \Omega)$ we have

\[ \int_{\Omega} g(\sigma) \, d\omega^p_\Omega \to \int_{\Omega} g(\sigma) \, d\omega^p_\Omega. \]

**Proof.** It is enough to show that, for every $\varphi \in C^\infty(\partial \Omega)$, the solutions of the problems $L^h u^h = 0$ in $\Omega$, $u^h = \varphi$ on $\partial \Omega$ $(h = 1, 2, \ldots)$ converge in $P$ to the solution $u$ of the problem $L u = 0$ in $\Omega$, $u = \varphi$ on $\partial \Omega$. Since $\varphi$ is a smooth function, $u_h$ is an equibounded sequence in $W^{2,p}(\Omega)$ for every $p < \infty$ and therefore converges weakly in $W^{2,p}(\Omega)$ and strongly in $L^\infty(\Omega)$ to some function $u \in W^{2,p}(\Omega)$. Obviously $u$ is the solution of the problem $L u = 0$ in $\Omega$, $u = \varphi$ on $\partial \Omega$, and the proof is complete.

3. Construction of a singular elliptic measure. Suppose $(h_n)$ and $(k_n)$ are two increasing sequences of positive integers with $h_n \to \infty$ and $k_n \to \infty$. Let $\varphi_n(x) = 1 + (1/2n^{1/2}) \cos(h_n x)$ and denote by $\psi$ a $C^\infty(\mathbb{R})$ function such that $\psi(t) = \psi(-t)$, $0 < \psi < 1$, $\psi = 1$ if $|t| < 1$, $\psi = 0$ if $|t| = 2$. Consider now the function $\alpha(x,y)$ defined by

\[\alpha(x,y) = \begin{cases} \varphi_n(x) & \text{if } |y| > 1/k_n, \\ \psi(k_{n+1}y)\varphi_{n+1}(x) + \left[ 1 - \psi(k_{n+1}y) \right] \varphi_n(x) & \text{if } 1/k_{n+1} < |y| < 1/k_n, \\ 1 & \text{if } y = 0. \end{cases}\]

If $k_{n+1} > 2k_n$, the function $\alpha$ is continuous in $R^2$ and $C^\infty$ except on the $x$-axis; moreover $\frac{1}{2} < \alpha < \frac{3}{2}$.
Let $B$ be as in §2 and $L = D_{xx}^2 + aD_{yy}^2$.

**Theorem.** If $\{h_n\}$ and $\{k_n\}$ are suitably chosen, the $L$-elliptic measure $\omega^p_L$ on $\partial B$ (evaluated at the point $P \in B$) is not absolutely continuous with respect to the Lebesgue measure on $[-1, 1]$.

**Remark.** Since the $L$-elliptic measures evaluated at different points of $B$ are mutually absolutely continuous (by maximum principle), the choice of the point $P$ is irrelevant.

**Proof of the Theorem.** We prove the theorem via an approximation argument. Observe that $\alpha$ is the uniform (in $R^2$) limit of the sequence of the $C^\infty$ functions defined by

$$
\alpha_n(x, y) = \begin{cases} 
\alpha(x, y) & \text{if } |y| > 1/k_n, \\
\varphi_n(x) & \text{if } |y| < 1/k_n.
\end{cases}
$$

As usual we denote by $L^n$ the operator $D_{xx}^2 + \alpha_nD_{yy}^2$ and by $\omega^p_L$ its elliptic measure on $\partial B$ evaluated at $P$.

Note that on $B_0$ (i.e. $B \cap \{y = 0\}$) the density of $\omega^p_L$ is given by $\varphi_n(x)D_yG^n(P; x, 0)$, where $G^n(P; x, y)$ denotes the Green's function of $L^n$.

Applying Lemma 4, we see that $\omega^p_L$ converges weakly to $\omega^p_L$; therefore the theorem will be proved if we choose $\{h_n\}$ and $\{k_n\}$ such that $\varphi_n(x)D_yG^n(P; x, 0)$ converges weakly to a singular measure on $[-1, 1]$.

We proceed by induction.

Set $h_1 = k_1 = 1$ and suppose we have already chosen $h_2, \ldots, h_n; k_2, \ldots, k_n$ in such a way that $h_j > 4h_{j-1}, k_j > 2k_{j-1}$ for $j = 1, 2, \ldots, n$. To choose $k_{n+1}$, put $c = \min_{x \in [-1, 1]} D_yG(x, 0)$, which is a positive number by Hopf's lemma (see [5, p. 65]) and define

$$
\tilde{\alpha}_n(x, y) = \begin{cases} 
\alpha_n(x, y) & \text{if } |y| > 1/k_n, \\
\psi(k_ny) + [1 - \psi(k_ny)]\varphi_n(x) & \text{if } |y| < 1/k_n.
\end{cases}
$$

If $k > 2k_n$, it is easy to check that $\tilde{\alpha}_n \in C^\infty(R^2)$ and moreover $\tilde{\alpha}_n$ converges to $\alpha_n$ in $L^2_{loc}(R^2)$ as $k \to \infty$ and $D_x\tilde{\alpha}_n, D_{xx}\tilde{\alpha}_n$ are equibounded in $R^2$. Let us denote by $\tilde{G}(P; x, y)$ the Green's function in $B$ for the operator $D_{xx}^2 + \tilde{\alpha}_nD_{yy}^2$ with pole $P$. Lemma 3 guarantees now the existence of an index $k_{n+1}$ such that $k_{n+1} > 2k_n$ and

$$
\max_{x \in [-1, 1]} |D_y(\tilde{\alpha}_{k_{n+1}} - \tilde{\alpha}_{k_n})(x, 0) - \tilde{D}_y(\alpha_n)(x, 0)| < \frac{c}{4n+2}.
$$

That is, since $\tilde{\alpha}_{k_{n+1}} = 1$ near $\{y = 0\}$ and $\tilde{G}^{k_{n+1}} = G^n = 0$ on $\{y = 0\}$,

$$
(3.1) \quad \max_{x \in [-1, 1]} |D_yG^{k_{n+1}}(P; x, 0) - \varphi_n(x)D_yG^n(P; x, 0)| < \frac{c}{4n+2}.
$$

To choose $h_{n+1}$ we define

$$
\tilde{\alpha}_n(x, y) = \begin{cases} 
\alpha_n(x, y) & \text{if } |y| > 1/k_n, \\
\psi(k_{n+1}y) + \frac{1}{2n + 1/2} \cos(hx) + [1 - \psi(k_{n+1}y)]\varphi_n(x) & \text{if } |y| < 1/k_n.
\end{cases}
$$
Clearly $\tilde{\alpha}_h \in C^\infty(R^2)$ and $\tilde{\alpha}_h \to \tilde{\alpha}_{h_n^+}$ weakly in $L^2(B)$ as $h \to \infty$; furthermore, $D_x \tilde{\alpha}_h$ and $D^2_{xx} \tilde{\alpha}_h$ are equibounded in $B$. Therefore, Lemma 1 implies the existence of an index $h_{n+1}$ such that $h_{n+1} > 4h_n$ and

$$\max_{x \in [-1,1]} |D_x \tilde{G}^{h_{n+1}}(P; x, 0) - D_x \tilde{G}^{h_n}(P; x, 0)| < \frac{c}{4^{n+2}},$$

where $\tilde{G}^{h}(P; x, 0)$ is the Green’s function in $B$ for the operator $D^2_{xx} + \tilde{\alpha}_h D^2_y$ with pole $P$.

By this choice of $h_{n+1}$ we have $\tilde{\alpha}_{h_n^+} = \alpha_{n+1}$; (3.1) and (3.2) give

$$\max_{x \in [-1,1]} |D_y G^{n+1}(P; x, 0) - D_y G^n(P; x, 0)\varphi_n(x)| < \frac{c}{4^{n+1}}.$$

From (3.3) we deduce that

$$q_{n+1} D_y G^{n+1}(P; x, 0)$$

$$= \left( \prod_{j=1}^{n} \varphi_j(x) \right) \left[ D_y G^1(P; x, 0) + \sum_{j=1}^{n} R_j(x) \left( \prod_{h=1}^{j} \varphi_h(x) \right)^{-1} \right]$$

where $\max_{x \in [-1,1]} |R_j(x)| < c/4^{j+1}$.

Since $\prod_{h=1}^{n} \varphi_h > 2^j$ it follows that the function between square brackets in (3.4) converges uniformly in $[-1, 1]$ to some continuous function $w(x)$; moreover $w(x) > \frac{3}{4} c > 0$. On the other hand, $\prod_{j=1}^{n} \varphi_j$ converges (see [8, vol. 1, p. 209]) weakly in the sense of the measures on $[-1, 1]$ to a singular measure.

We conclude that $\omega_L^P$ also converges weakly to a singular measure on $[-1, 1]$ and so the proof is complete.

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