A NONVARIATIONAL SECOND ORDER ELLIPTIC OPERATOR WITH SINGULAR ELLIPTIC MEASURE

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Abstract. We exhibit an example which proves that the elliptic measure for a second-order operator of the form $\sum_{i,j=1}^{n} a_{ij}D^2_{x_i x_j}$ with continuous coefficients can be singular with respect to the surface measure on the boundary of a smooth two-dimensional domain.

1. Introduction. In the papers [2] and [4], examples of singular elliptic measures for second-order operators in divergence form are given. In this note, following the ideas contained in [4], we exhibit an analogous example in the nonvariational case.

We recall the definition of elliptic measure. Let $\Omega$ be a bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and $L = \sum_{i,j=1}^{n} a_{ij}D^2_{x_i x_j}$, a uniformly elliptic operator with continuous coefficients in $\overline{\Omega}$. It is well known that, for every $g \in C(\partial \Omega)$, there exists a unique solution $u \in W^{2,2}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of the problem

$$Lu = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega.$$

The classical maximum principle and the Riesz representation theorem imply that for each $P \in \Omega$ there exists a Borel measure on $\partial \Omega$, $\omega_L^P$ (the $L$-elliptic measure evaluated at $P$), such that the following formula holds:

$$u(P) = \int_{\partial \Omega} g(\sigma) \, d\omega_L^P(\sigma).$$

On the other hand, by a result of Pucci and Alexandrov (see [1] and [6]), the solution, vanishing on the boundary, of the equation $Lv = f$, $f \in L^n(\Omega)$, satisfies the following estimate

$$\max_{P \in \overline{\Omega}} |v(P)| \leq c\|f\|_{L^n(\Omega)} \quad (1.1)$$

where the constant $c$ depends only on the ellipticity constants and the geometry of $\Omega$.

Notice that, by a result of Talenti [7], in dimension two we have the stronger inequality

$$\|v\|_{W^{2,2}(\Omega)} \leq c\|f\|_{L^2(\Omega)}, \quad (1.2)$$

where $c$ depends on the same parameters as before.

Pucci-Alexandrov's theorem implies the existence of the Green's function $G(P; Q)$, such that $G(P; \cdot) \in L^{n/(n-1)}(\Omega)$ for every fixed $P$ and $v(P) = \int_{\Omega} G(P; Q)f(Q) \, dQ$.
In the case of smooth coefficients, the divergence theorem gives the connection between the Green's function and the $L$-elliptic measure; we have
\[
dw^P(\sigma) = \sum_{i,j=1}^n a_j(\sigma) v_i D_{ij} G(P; \sigma) \, d\sigma
\]
where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the inward unit normal to $\partial \Omega$, and $d\sigma$ is the usual $(n - 1)$-dimensional surface measure on $\partial \Omega$.

Furthermore we recall that in $\Omega \setminus \{ P \}$, $G(P, \cdot)$ satisfies the equation $L^* G(P; \cdot) = 0$, where $L^*$ is the adjoint operator of $L$; formally $L^* u = \sum_{i,j=1}^n D^2_{ij}(a_j u)$. 

2. Main lemmas. In this section, $B$ is a bounded $C^\infty$ domain in the upper half-plane $R^2_+$ adjacent to the $x$-axis such that
\[
B_0 = \{ (x, y) : |x| < 2, y = 0 \} \subseteq \partial B.
\]
$P$ is a given point in $B$.

**Lemma 1.** Suppose $\beta = \beta(x,y)$, $\beta^h = \beta^h(x,y)$ ($h = 1, 2, \ldots$) are $C^\infty$ functions in $R^2$ satisfying the following conditions:

(a) $0 < \frac{1}{2} \leq \beta \leq \frac{3}{2}, \frac{1}{2} \leq \beta^h < \frac{3}{2}, \forall h$;

(b) $|D_y \beta^h(x,y)| < c_1, |D^2_{yy} \beta^h(x,y)| < c_2, \forall (x,y) \in B, \forall h$;

(c) $\beta^h$ converges weakly in $L^2(B)$ to $\beta$, as $h \to \infty$;

(d) $\beta^h(x,y) = \beta^h(x,-y)$.

Denote by $E$ and $E_h$ respectively the operators $D^2_{xx} + \beta D^2_{yy}$ and $D^2_{xx} + \beta^h D^2_{yy}$.

Claim. If $G(x,y) = G(P; x,y)$ and $G^h(x,y) = G(P; x,y)$ are the Green's functions for $E$ and $E_h$ in $B$ with pole $P$, then
\[
D_y G^h(x, 0) \to D_y G(x, 0) \quad \text{uniformly for } x \in [-1, 1].
\]

**Proof.** Denote by $T_+$ the set $\{(x, y) : |x| < 1 + \varepsilon, 0 < y < \mu \}$ with $\varepsilon, \mu$ small enough to ensure $\overline{T}_+ \subseteq B$, $P \notin \overline{T}_+$. $T_-$ will be the set $\{(x, y) : |x| < 1 + \varepsilon, -\mu < y < 0 \}$ and $T = T_+ \cup T_-$. In $T_+$ we have
\[
E_h^* G = D^2_{xx} G^h + D^2_{yy} (\beta^h G^h) = 0.
\]
Furthermore $G^h(x, 0) = 0$ if $|x| < 2$.

The equation (2.1) can be written in the following divergence form:
\[
D^2_{xx} G^h + D_y (\beta^h D_y G^h) = -D_y (D_y \beta^h G^h).
\]

Extend now $G^h(P; x, y)$ to an odd function with respect to $y$ across $y = 0$ and call $\tilde{G}^h(P; x, y)$ the extended function; then $\tilde{G}^h(P; x, y)$ satisfies in $T$ the equation (2.2). On the other hand, by (1.1), it is easy to show that $\tilde{G}^h(P; x, y)$ is equibounded in $L^2(T)$ and, by the hypothesis (b) we have $\|D_y \beta^h\|_{L^\infty(T)} < c_1$.

Well-known results on divergence form equations imply that $\|\tilde{G}^h(P; \cdot)\|_{W^{1,2}(T)} < c$ (independent of $h$) and thus, by Sobolev's immersion theorem, $\tilde{G}^h(P; \cdot)$ is equibounded in $L^p(T)$ for every $p > 2$.

Therefore, Meyers' theorem (see [3]) implies the existence of $\delta > 0$ such that
\[
\|\tilde{G}^h(P; \cdot)\|_{W^{1,2}(T)} < \text{const} \quad \text{(independent of h)}.
\]
In particular we have
\begin{equation}
\|G^h(P; \cdot)\|_{\mathcal{W}^{1,2+\epsilon}(T_+)} < \text{const} \quad \text{(independent of } h). 
\end{equation}

Differentiating now (2.1) with respect to $y$ and putting $v^h = D_y G^h$, we see that $v^h$ satisfies in $T_+$ the following divergence form equation:

$$
D_{xx}v^h + D_y(\beta^h D_y v^h) = -2D_y(D_y \beta^h v^h) - D_y(D_{yy}^h \beta^h G^h).
$$

By (2.3) and hypothesis (b), the right-hand side is the divergence of an equibounded (in $L^{2+\delta}(T_+)$) vector field. Using once more Meyers' result we deduce that

\begin{equation}
\|v^h\|_{\mathcal{W}^{1,2+\epsilon}(T_+)} < \text{const} \quad \text{(independent of } h).
\end{equation}

From (2.1) and (2.4) it follows that $\|G^h(P; \cdot)\|_{\mathcal{W}^{1,2+\epsilon}(T_+)} < \text{const}$ (independent of $h$). Sobolev's immersion theorem implies now that $G^h$ (actually a subsequence) converges in $C^1(\overline{T}_+)$ to some continuous function $g(x,y)$. The conclusion of Lemma 1 will follow from the following lemma.

**Lemma 2.** Under the hypothesis of Lemma 1, $G^h(P; x, y)$ converges weakly in $L^2(B)$ to $G(P; x, y)$.

**Proof.** Consider the function

\begin{equation}
u^h(P) = \int_B G^h(P; x, y)f(x, y) \, dx \, dy
\end{equation}

where $f \in L^2(B)$.

The function $u^h(P)$ is the solution, vanishing on $\partial B$, of $E_u = f$ in $B$. From the result of Talenti [7], we have $\|u^h\|_{\mathcal{W}^{2,2}(B)} < c\|f\|_{L^2(B)}$ with $c$ depending only on the geometry of $B$. Therefore $u^h$ admits a subsequence converging weakly in $W^{2,2}(B)$ and strongly in $W^{1,p}(B)$ for every $p > 2$ to a function $u \in W^{2,2}(B)$, which vanishes on $\partial B$. We will show that $u$ is the solution, vanishing on $\partial B$, of $Eu = f$.

We have $D_{xx}^2 u^h + \beta^h D_{yy}^2 u^h = f$, and therefore

$$
D_{xx}^2 u^h + D_y(\beta^h D_y u^h) = D_y \beta^h D_y u^h + f.
$$

It is enough to show that $(D_y \beta^h)(D_y u^h)$ converges weakly in $L^2(B)$ to $(D_y \beta)(D_y u)$ and this is an easy consequence of the following facts: $\beta^h \to \beta$ in $L^2(B)$ weakly, $\|\beta^h\|_{L^\infty(B)} < \text{const}$, $D_y u^h \to D_y u$ in $L^2(B)$ strongly.

On the other hand $G^h(P; \cdot)$ is equibounded in $L^2(B)$ and so has a subsequence which converges weakly in $L^2(B)$ to some function $v_p \in L^2(B)$. From the representation formula (2.5), letting $h_n$ tend to infinity, we have

\begin{equation}
\nu(P) = \int_B v_p(x, y)f(x, y) \, dx \, dy.
\end{equation}

Since (2.6) holds for any $f \in L^2(B)$, we conclude $v_p = G$.

**Lemma 3.** Suppose $\beta$ and $\beta^h$ are $C^\infty$ functions in $R^2$ satisfying the following hypotheses:

(a) $\frac{1}{2} < \beta < \frac{3}{2}$, $\frac{1}{2} < \beta^h < \frac{3}{2}$ $\forall h$;

(b) $|D_x \beta^h(x, y)| < C_1$, $|D_{xx}^2 \beta^h(x, y)| < C_2$ $\forall (x, y) \in B$, $\forall h$;
(c) $\beta^h$ converges to $\beta$ in $L^2_{\text{loc}}(B)$;
(d) $\beta^h(x,y) = \beta^h(x,-y)$.

Denote by $E$ and $E_h$ the operators $D_{xx}^2 + \beta \cdot D_{yy}^2$ and $D_{xx}^2 + \beta^h \cdot D_{yy}^2$ and by $G(x,y) = G(P; x, y)$, $G^h(x, y) = G^h(P; x, y)$ their Green’s functions in $B$ with pole $P$.

**Claim.** $D_y(\beta^h G^h)(x, 0) \to D_y(\beta G)(x, 0)$ uniformly for $x \in [-1, 1]$.

**Proof.** Let $T_+$ be as in the proof of Lemma 1. In $T_+$ we have

$$D_{xx}^2 G^h + D_{yy}^2(\beta^h G^h) = 0.$$  

This means that the function $v^h = \beta^h G^h$ satisfies the equation $D_{xx}^2(v^h/\beta^h) + D_{yy}^2 v^h = 0$. Furthermore $v^h(x, y) = 0$ if $|x| < 1 + \varepsilon$ and $y = 0$. Arguing now as in Lemma 1, having interchanged the roles of $x$ and $y$, we deduce that $v^h \to v$ in $C^1(\overline{T}_+)$ where $v$ is some $C^1$ function in $\overline{T}_+$.

The lemma will be proved if we show that $D_y v(x, 0) = D_y(\beta G)(x, 0)$ for $|x| < 1$. But, if we recall that the $L$-elliptic measure $\omega_L$ has density $D_y(\beta G)(x, 0)$, this follows from the following general lemma.

**Lemma 4.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ and $L^h = \sum_{i,j=1}^{n} a^h_{ij} D_{ij}$ ($h = 1, 2, \ldots$) with $a^h_{ij} \in C(\overline{\Omega})$. Suppose furthermore that

(a) for every $\xi \in \mathbb{R}^n$, $|\xi|^2 \leq \sum_{i,j=1}^{n} a^{h}_{ij} \xi_i \xi_j < M|\xi|^2$ with $\mu$ and $M$ independent of $h$;
(b) $a^h_{ij} \to a_{ij}$ in $L^2_{\text{loc}}(\Omega)$.

Then, if $L$ is the limit operator $\sum_{i,j=1}^{n} a_{ij} D_{ij}$ and $P$ is fixed in $\Omega$, $\omega^p_L$ converges weakly to $\omega^p_L$, that is, for every $g \in C(\partial \Omega)$ we have

$$\int_{\Omega} g(\sigma) \, d\omega^p_L \to \int_{\Omega} g(\sigma) \, d\omega^p_L.$$  

**Proof.** It is enough to show that, for every $\varphi \in C^\infty(\partial \Omega)$, the solutions of the problems $L^h u^h = 0$ in $\Omega$, $u^h = \varphi$ on $\partial \Omega$ ($h = 1, 2, \ldots$) converge in $P$ to the solution $u$ of the problem $Lu = 0$ in $\Omega$, $u = \varphi$ on $\partial \Omega$.

Since $\varphi$ is a smooth function, $u_h$ is an equibounded sequence in $W^{2,p}(\Omega)$ for every $p < \infty$ and therefore converges weakly in $W^{2,p}(\Omega)$ and strongly in $L^\infty(\Omega)$ to some function $u \in W^{2,p}(\Omega)$. Obviously $u$ is the solution of the problem $Lu = 0$ in $\Omega$, $u = \varphi$ on $\partial \Omega$, and the proof is complete.

3. Construction of a singular elliptic measure. Suppose $\{h_n\}$ and $\{k_n\}$ are two increasing sequences of positive integers with $h_n \to \infty$ and $k_n \to \infty$. Let $\varphi_n(x) = 1 + (1/2n^{1/2}) \cos(h_n x)$ and denote by $\psi$ a $C^\infty_0(\mathbb{R})$ function such that $\psi(t) = \psi(-t)$, $0 < \psi < 1$, $\psi = 1$ if $|t| < 1$, $\psi = 0$ if $|t| > 2$. Consider now the function $\alpha(x,y)$ defined by

$$\alpha(x,y) = \begin{cases} \varphi_n(x) & \text{if } |y| > 1/k_n, \\ \psi(k_{n+1} y) \varphi_{n+1}(x) + [1 - \psi(k_{n+1} y)] \varphi_n(x) & \text{if } 1/k_{n+1} < |y| < 1/k_n, \\ 1 & \text{if } y = 0. \end{cases}$$

If $k_{n+1} > 2k_n$, the function $\alpha$ is continuous in $\mathbb{R}^2$ and $C^\infty$ except on the $x$-axis; moreover $\frac{1}{2} < \alpha < \frac{3}{2}$. 

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Let $B$ be as in §2 and $L = D_{xx}^2 + aD_{yy}^2$.

**Theorem.** If $\{h_n\}$ and $\{k_n\}$ are suitably chosen, the $L$-elliptic measure $\omega_L^n$ on $\partial B$ (evaluated at the point $P \in B$) is not absolutely continuous with respect to the Lebesgue measure on $[-1, 1]$.

**Remark.** Since the $L$-elliptic measures evaluated at different points of $B$ are mutually absolutely continuous (by maximum principle), the choice of the point $P$ is irrelevant.

**Proof of the Theorem.** We prove the theorem via an approximation argument. Observe that $\alpha$ is the uniform (in $\mathbb{R}^2$) limit of the sequence of the $C^\infty$ functions defined by

$$\alpha_n(x, y) = \begin{cases} 
\alpha(x, y) & \text{if } |y| > 1/k_n, \\
\psi(k_n) + 1/\sqrt{2(n + 1)} & \text{if } |y| < 1/k_n.
\end{cases}$$

As usual we denote by $L^n$ the operator $D_{xx}^2 + \alpha_n D_{yy}^2$ and by $\omega_L^n$ its elliptic measure on $\partial B$ evaluated at $P$.

Note that on $B_0$ (i.e. $\overline{B} \cap \{y = 0\}$) the density of $\omega_L^n$ is given by $\varphi_n(x)D_yG^n(P; x, 0)$, where $G^n(P; x, y)$ denotes the Green's function of $L^n$.

Applying Lemma 4, we see that $\omega_L^n$ converges weakly to $\omega_L^p$; therefore the theorem will be proved if we choose $\{h_n\}$ and $\{k_n\}$ such that $\varphi_n(x)D_yG^n(P; x, 0)$ converges weakly to a singular measure on $[-1, 1]$.

We proceed by induction.

Set $h_1 = k_1 = 1$ and suppose we have already chosen $h_2, \ldots, h_n, k_2, \ldots, k_n$ in such a way that $h_j > 4h_{j-1}, k_j > 2k_{j-1}$ for $j = 1, 2, \ldots, n$. To choose $k_{n+1}$, put $c = \min_{x \in [-1, 1]} D_yG^1(x, 0)$, which is a positive number by Hopf's lemma (see [5, p. 65]) and define

$$\tilde{\alpha}_k(x, y) = \begin{cases} 
\alpha_n(x, y) & \text{if } |y| > 1/k_n, \\
\psi(k_n) + 1/\sqrt{2(n + 1)} & \text{if } |y| < 1/k_n.
\end{cases}$$

If $k > 2k_n$, it is easy to check that $\tilde{\alpha}_k \in C^\infty(\mathbb{R}^2)$ and moreover $\tilde{\alpha}_k$ converges to $\alpha_n$ in $L^2_{\text{loc}}(\mathbb{R}^2)$ as $k \to \infty$ and $D_x\tilde{\alpha}_k, D_{xx}\tilde{\alpha}_k$ are equibounded in $\mathbb{R}^2$. Let us denote by $\tilde{G}^k(P; x, y)$ the Green's function in $B$ for the operator $D_{xx}^2 + \tilde{\alpha}_k D_{yy}^2$ with pole $P$.

Lemma 3 guarantees now the existence of an index $k_{n+1}$ such that $k_{n+1} > 2k_n$ and

$$\max_{x \in [-1, 1]} \left| D_y(\tilde{\alpha}_{k_{n+1}} \tilde{G}^{k_{n+1}})(x, 0) - D_y(\alpha_n G^n)(x, 0) \right| < \frac{c}{4^{n+2}}.$$ 

That is, since $\tilde{\alpha}_{k_{n+1}} = 1$ near $\{y = 0\}$ and $\tilde{G}^{k_{n+1}} = G^n = 0$ on $\{y = 0\}$,

$$\max_{x \in [-1, 1]} \left| D_y G^{k_{n+1}}(P; x, 0) - \varphi_n(x)D_yG^n(P; x, 0) \right| < \frac{c}{4^{n+2}}.$$ 

To choose $h_{n+1}$ we define

$$\tilde{\alpha}_h(x, y) = \begin{cases} 
\alpha_n(x, y) & \text{if } |y| > 1/k_n, \\
\psi(k_{n+1}y) \left[ 1 + \frac{1}{2(n + 1)^{1/2}} \cos(hx) \right] + [1 - \psi(k_{n+1}y)] \varphi_n(x) & \text{if } |y| < 1/k_n.
\end{cases}$$
Clearly $\alpha_h \in C^\infty(R^2)$ and $\alpha_h \to \alpha_{h^*}$ weakly in $L^2(B)$ as $h \to \infty$; furthermore, $D_y \alpha_h$ and $D_{yy}^2 \alpha_h$ are equibounded in $B$. Therefore, Lemma 1 implies the existence of an index $h_{n+1}$ such that $h_{n+1} > 4h_n$ and

$$\max_{x \in [-1,1]} |D_y \tilde{G}^{h_{n+1}}(P; x, 0) - D_y \tilde{G}^{h_n}(P; x, 0)| < \frac{c}{4^{n+2}},$$

where $\tilde{G}^h(P; x, 0)$ is the Green’s function in $B$ for the operator $D_{xx}^2 + \alpha_h D_{yy}^2$ with pole $P$.

By this choice of $h_{n+1}$ we have $\alpha_{h_{n+1}} = \alpha_{n+1}$; (3.1) and (3.2) give

$$\max_{x \in [-1,1]} |D_y G^{n+1}(P; x, 0) - D_y G^n(P; x, 0)\varphi_n(x)| < \frac{c}{4^{n+1}}.$$

From (3.3) we deduce that

$$q_{n+1} D_y G^{n+1}(P; x, 0)$$

(3.4)

$$= \left( \prod_{j=1}^{n} q_j(x) \right) \left[ D_y G^1(P; x, 0) + \sum_{j=1}^{n} R_j(x) \left( \prod_{h=1}^{j} q_h(x) \right)^{-1} \right]$$

where $\max_{x \in [-1,1]} |R_j(x)| < c/4^{j+1}$.

Since $\prod_{j=1}^{3} q_j > 2^3$ it follows that the function between square brackets in (3.4) converges uniformly in $[-1, 1]$ to some continuous function $w(x)$; moreover $w(x) > \frac{3}{4} c > 0$. On the other hand, $\prod_{j=1}^{3} q_j$ converges (see [8, vol. I, p. 209]) weakly in the sense of the measures on $[-1, 1]$ to a singular measure.

We conclude that $\omega_L^p$ also converges weakly to a singular measure on $[-1, 1]$ and so the proof is complete.

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