THE ASYMPTOTIC BEHAVIOR OF A CLASS OF
NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. Let $u'' + f(t, u) = 0$ be a nonlinear differential equation. If there are
two nonnegative continuous functions $\sigma(t)$, $\phi(t)$ for $t > 0$, and a continuous
function $g(u)$ for $u > 0$, such that (i) $\int_{t_0}^{\infty} \sigma(t)\phi(t) \, dt < \infty$; (ii) for $u > 0$, $g(u)$ is
positive and nondecreasing; (iii) $|f(t, u)| < \sigma(t)\phi(t)g(|u|/t)$ for $t > 1$, $-\infty < u < \infty$,
then the equation has solutions asymptotic to $a + bt$, where $a, b$ are constants
and $b \neq 0$. Our result generalizes a theorem of D. S. Cohen [3].

Consider the nonlinear differential equation

\[(1) \quad u'' + f(t, u) = 0.\]

D. S. Cohen [3] proved the following theorem.

**Theorem A.** Suppose $f(t, u)$ satisfies the following conditions:

(H-1) $f(t, u)$ is continuous on $D$: $t > 0$, $-\infty < u < \infty$.

(H-2) The derivative $f_u(t, u)$ exists on $D$ and $f_u(t, u) > 0$ on $D$.

(H-3) $f(t, u)| < f_u(t, 0)|u|$ on $D$.

In addition, suppose that

\[(2) \quad \int_{1}^{\infty} f_u(t, 0) \, dt < \infty.\]

Then equation (1) has solutions which are asymptotic to $a + bt$ as $t \to \infty$, where $a, b$
are constants and $b \neq 0$.

In the proof of Theorem A, Cohen used R. Bellman’s method [1, pp. 114–115]
and Gronwall’s inequality. In this paper we use the same method and Bihari’s
inequality [2] to generalize Theorem A.

**Theorem B.** Let $f(t, u)$ be continuous on $D$: $t > 0$, $-\infty < u < \infty$. If there are two
nonnegative continuous functions $\sigma(t)$, $\phi(t)$ for $t > 0$, and a continuous function $g(u)$
for $u > 0$, such that

(i) $\int_{1}^{\infty} \sigma(t)\phi(t) \, dt < \infty$,

(ii) for $u > 0$, $g(u)$ is positive and nondecreasing,

(iii) $|f(t, u)| < \sigma(t)\phi(t)g(|u|/t)$ for $t > 1$, $-\infty < u < \infty$,

then the equation (1) has solutions which are asymptotic to $a + bt$, where $a, b$
are constants and $b \neq 0$. 

Received by the editors August 11, 1980 and, in revised form, April 15, 1981.


Key words and phrases. Gronwall’s inequality, Bihari’s inequality.

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0002-9939/82/0000-0416/$01.50

235

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Remark. If we let \( v(t) = f_u(t, 0) \), \( q(t) = t \), \( g(u) = u \) in Theorem B, we obtain Theorem A.

Proof of Theorem B. Integrating (1) twice on \([1, t]\), we have

\[
u(t) = c_1 + c_2 t - \int_1^t (t - s) f(s, u(s)) \, ds.
\]

Choose \( c_1 > 1 \) and let \( c_3 = c_1 + |c_2| \). Then for \( t > 1 \) we have

\[
\frac{|u(t)|}{t} \leq c_3 + \int_1^t f(s, u(s)) \, ds,
\]

\[
\leq c_3 + \int_1^t v(s) \varphi(s) g(|u(s)|/s) \, ds.
\]

By Bihari's inequality we have

\[
\frac{|u(t)|}{t} \leq G^{-1} \left( G(c_3) + \int_1^t v(s) \varphi(s) \, ds \right).
\]

Here \( G(x) = \int_1^x dt / g(t) \), \( G^{-1}(x) \) is the inverse function of \( G(x) \). From \( g(t) > 0 \) we know that \( G(x) \) is increasing; hence \( G^{-1}(x) \) exists, and is also increasing.

Now let \( c_4 = G(c_3) + \int_1^t v(s) \varphi(s) \, ds \). Since \( G^{-1}(x) \) is increasing, we have

\[
\frac{|u(t)|}{t} \leq G^{-1}(c_4).
\]

Differentiating (3), we have

\[
u'(t) = c_2 - \int_1^t f(s, u(s)) \, ds.
\]

By conditions (i), (ii), (iii) and (5), we have

\[
\int_1^t |f(s, u(s))| \, ds \leq \int_1^t v(t) \varphi(t) g(|u(s)|/s) \, ds
\]

\[
\leq g(G^{-1}(c_4)) \int_1^t v(s) \varphi(s) \, ds < \infty.
\]

Therefore \( u'(t) \to c_2 - \int_1^\infty f(s, u(s)) \, ds \) as \( t \to \infty \). If we choose \( c_2 \) sufficiently large, then \( u'(t) > 1 \). Hence \( \lim_{t \to \infty} u'(t) \neq 0 \). This proves the theorem.

We give an example to which Theorem B applies but Theorem A does not.

Example.

\[
u'' + t^{-4} u^2 \cos u = 0.
\]

Since \( f_u(t, 0) = 0 \), (H-3) does not hold and Theorem A does not apply. Let \( v(t) = t^{-4} \), \( q(t) = t^2 \), \( g(u) = u^2 \). Then conditions (i), (ii) and (iii) are satisfied and equation (7) has solutions asymptotic to \( a + bt \) as \( t \to \infty \).

Acknowledgement. The author is indebted to the referee for simplifying the conditions of Theorem B.

References


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