TOTALLY REAL MINIMAL IMMERSIONS
OF n-DIMENSIONAL REAL SPACE FORMS
INTO n-DIMENSIONAL COMPLEX SPACE FORMS

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Abstract. n-dimensional totally real minimal submanifolds of constant sectional curvature in n-dimensional complex space forms are totally geodesic or flat.

1. Introduction. B. Y. Chen and K. Ogiue [1] showed that an n-dimensional, totally real, minimal submanifold of constant curvature c in an n-dimensional complex space form is totally geodesic or \( c < 0 \). On the other hand, [2, Theorem 7] implies that a complete totally real minimal surface of constant sectional curvature in a 2-dimensional complex space form is totally geodesic or flat. We shall prove a generalization of these results.

Theorem. Let \( M \) be an n-dimensional, totally real, minimal submanifold of constant sectional curvature c, immersed in an n-dimensional complex space form. Then \( M \) is totally geodesic or flat \( (c = 0) \).

2. Preliminary. We denote by \( M^n(4\hat{c}) \) an n-dimensional complex space form of constant holomorphic sectional curvature \( 4\hat{c} \) with complex structure \( J \) and metric \( \bar{g} \). Let \( M \) be an n-dimensional Riemannian manifold of constant sectional curvature \( c \) isometrically immersed in \( M^n(4\hat{c}) \) as a totally real submanifold. We denote by \( \sigma \) the second fundamental form of the immersion

\[
\sigma(X, Y) = \bar{\nabla}_X Y - \nabla_X Y,
\]

where \( \bar{\nabla} \) (resp. \( \nabla \)) is the covariant differentiation with respect to \( \bar{g} \) (resp. the metric \( g \) of \( M \)).

We put \( T = -J\sigma \). Then \( T \) is a symmetric tensor field of type \((1, 2)\) on \( M \) and it satisfies

\[
(2.1) \quad g(T(X, Y), Z) = g(T(X, Z), Y).
\]

Moreover, the equations of Gauss, Ricci and Codazzi are given respectively by [1],

\[
(\hat{c} - c)\{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \} + g(T(X, Z), T(Y, W)) - g(T(X, W), T(Y, Z)) = 0 \quad \text{(the equations of Gauss and Ricci),}
\]

\[
(2.2) \quad (\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z) = 0 \quad \text{(the equation of Codazzi)}.
\]

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3. A lemma. In this section, we prove the following.

**Lemma.** Let $T$ be a symmetric 3-linear map of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$ such that

$$A \{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \} + \sum_{m=1}^{n} T(X, Z, f_m)T(Y, W, f_m)$$

$$- \sum_{m=1}^{n} T(X, W, f_m)T(Y, Z, f_m) = 0 \quad \text{and} \quad A > 0,$$

(3.1)

$$\sum_{m=1}^{n} T(X, f_m, f_m) = 0,$$

(3.2)

where $g$ is the Euclidean metric of $\mathbb{R}^n$ and $f_1, \ldots, f_n$ is an orthonormal basis. If we choose an orthonormal basis $e_1, \ldots, e_n$, such that each $e_i$ is a maximum point of the cubic function $T(X, X, X)$, restricted to $\{ X \in \mathbb{R}^n : \|X\| = 1, \text{and } X \text{ is orthogonal to } e_1, \ldots, e_{i-1} \}$, then $T$ has the following expression:

$$T(e_a, e_a, e_a) = (n - a)\sqrt{\frac{A}{(n - a + 1)} + \cdots + \frac{A}{(n - a + 1) \cdots n}},$$

$$T(e_a, e_b, e_c) = -\sqrt{\frac{A}{(n - a + 1)} + \cdots + \frac{A}{(n - a + 1) \cdots n}} \delta_{ab},$$

where $1 < a < n$ and $a < i, j, k < n$ unless $i_a = j_a = a$.

**Proof.** If $A = 0$, the assumption (3.1) and (3.2) imply $T = 0$. Hence we may consider the case $A > 0$. We shall prove this lemma by induction on the dimension of $\mathbb{R}^n$. It is easy to prove that $T(e_1, e_1, X) = 0$ for all $X$ orthogonal to $e_1$. Since $T(e_1, X, Y)$ is symmetric with respect to $X$ and $Y$, we can choose an orthonormal basis $f_1(= e_1), f_2, \ldots, f_n$ which satisfies $T(f_1, f_1, f_1) = A_1 \delta_{11}$. Using (3.1) and (3.2), we obtain

$$\lambda_1 > 0,$$

(3.3)

$$\lambda_1 + \cdots + \lambda_n = 0,$$

(3.4)

$$A + \lambda_1 \lambda_2 - (\lambda_a)^2 = 0 \quad \text{for } 1 < a < n.$$

(3.5)

If $n = 2$, the result follows from (3.3), (3.4) and (3.5).

Assume that the lemma is true for $< n - 1$ and consider the lemma for $\mathbb{R}^n$. Let $f_1, \ldots, f_n$ be the orthogonal basis chosen above. From (3.5), we must consider two cases.

**Case 1.** $\lambda_2 = \cdots = \lambda_{p+1} (= \mu)$ and $\lambda_{p+2} = \cdots = \lambda_n (= \nu)$, where $\mu \neq \nu$ and $1 < p < n - 2$.

**Case 2.** $\lambda_2 = \cdots = \lambda_n (= \mu)$.

If Case 1 holds, then, without loss of generality, we may assume $2p < n - 1$. From (3.4) and (3.5), it follows that

$$\mu^2 = (n - p)A / (p + 1), \quad \nu^2 = (p + 1)A / (n - p),$$

$$\lambda_1 \mu = (n - 2p - 1)A / (p + 1) \quad \text{and} \quad \lambda_1 \nu = -(n - 2p - 1)A / (n - p).$$
Thus we have \( n - 2p - 1 > 0 \) and hence, \( n > 3 \),

\[
\mu = \sqrt{(n - p)A / (p + 1)} , \quad \gamma = -\sqrt{(p + 1)A / (n - p)} ,
\]

\[
\lambda_1 = \sqrt{(n - p)A / (p + 1)} - \sqrt{(p + 1)A / (n - p)} .
\]

Therefore we use the following convention on the ranges of indices: \( a < a' < p + 1 < a'' < n \). Using (3.1), we have \((\lambda_a - \lambda_{a''})T(f_a, f_a', f_{a''}) = 0\), which, together with \(\lambda_a - \lambda_{a''} \neq 0\), implies \(T(f_a, f_a', f_{a''}) = 0\). Let \( N' \) (resp. \( N'' \)) be the linear subspace of \( \mathbb{R}^n \) spanned by \( f_2, \ldots, f_n \) (resp. \( f_2, \ldots, f_{p+1}; f_{p+2}, \ldots, f_n \)). Then we obtain

\[
T(X', Y', Z'') = 0 , \quad T(f_1, X', Y') = \sqrt{(n - p)A / (p + 1)} g(X', Y') ,
\]

\[
T( f_1, X'', Y'') = -\sqrt{(p + 1)A / (n - p)} g(X'', Y'') , \quad T(f_1, X', Y'') = 0 ,
\]

where \( X \in N, X', Y' \in N' \) and \( X'', Y'', Z'' \in N'' \), which, together with (3.1) and (3.2), imply that

\[
A(n + p) / (p + 1) \{ g(X', Z')g(Y', W') - g(X', W')g(Y', Z') \}
\]

\[
+ \sum_{a'=2}^{p+1} T(X', Z', f_{a'}) T(Y', W', f_{a'}) - \sum_{a'=2}^{p+1} T(X', W', f_{a'}) T(Y', Z', f_{a'}) = 0 ,
\]

\[
\sum_{a'=2}^{p+1} T(X', f_{a'}, f_{a''}) = 0
\]

and

\[
A(n + 1) / (n - p) \{ g(X'', Z'')g(Y'', W'') - g(X'', W'')g(Y'', Z'') \}
\]

\[
+ \sum_{a''=p+2}^{n} T(X'', Z'', f_{a''}) T(Y'', W'', f_{a''})
\]

\[
- \sum_{a''=p+2}^{n} T(X'', W'', f_{a''}) T(Y'', Z'', f_{a''}) = 0 ,
\]

\[
\sum_{a''=p+2}^{n} T(X'', f_{a''}, f_{a'}) = 0 ,
\]

where \( X', Y', Z', W' \in N' \) and \( X'', Y'', Z'', W'' \in N'' \). Since the dimensions of \( N' \) and \( N'' \) are less than \( n \), from the assumption we obtain unit vectors \( e' \in N' \) and \( e'' \in N'' \) such that

\[
T(e', e', e') = (p - 1)\sqrt{A(n + 1)/p(p + 1)} ,
\]

\[
T(e'', e'', e'') = (n - p - 2)\sqrt{A(n + 1)/(n - p - 1)(n - p)} .
\]

Therefore the definition of \( e_1 \) gives

\[
\sqrt{A(n - p) / (p + 1)} - \sqrt{A(p + 1) / (n - p)} > \text{Max}\{ (p - 1)\sqrt{A(n + 1)/p(p + 1)} ,
\]

\[
(n - p - 2)\sqrt{A(n + 1)/(n - p - 1)(n - p)} .
\]
which implies \( \sqrt{(n - p)/(p + 1)} > (p - 1)\sqrt{(n + 1)/p(p + 1)} \). We immediately obtain \( p = 1 \) or \( 2 \). This, together with the inequality, induces a contradiction for \( n > 3 \). It is easy to treat Case 2 by the same argument as Case 1. Q.E.D.

4. **Proof of Theorem.** Let \( T \) be the second fundamental form of the immersion as a symmetric bilinear map \( TM \times TM \) into \( TM \). By (2.1), we may consider \( T \) as a symmetric 3-linear map of \( TM \times TM \times TM \) into \( R \). By (2.2) and the minimality of \( M \), it satisfies (3.1) and (3.2) for \( A = \bar{c} - c \). We may assume that \( M \) is not totally geodesic, i.e., \( A \neq 0 \). We easily obtain a local field of orthonormal frames \( e_1, \ldots, e_n \) such that the lemma holds. We denote by \( \omega_{ij} \) the Levi-Civita connection with respect to \( e_1, \ldots, e_n \). Using (2.3), we have

\[
-\sqrt{\frac{A}{n}} \sum_{i=1}^{n} \omega_{ij}(e_i) e_i - \sum_{i=1}^{n} \omega_{ij}(e_i) T(e_j, e_i) - \sum_{i=1}^{n} \omega_{ij}(e_i) T(e_i, e_i)
- (n - 1)\sqrt{\frac{A}{n}} \sum_{i=1}^{n} \omega_{ij}(e_i) e_i + 2 \sum_{i=1}^{n} \omega_{ij}(e_i) T(e_j, e_i) = 0, \quad \text{for all } a \neq 1.
\]

Taking the innerproduct of it and \( e_1 \), we obtain \( \omega_{ij}(e_i) = 0 \). This, together with the innerproduct of the above and \( e_b \) \((b \neq 1)\), implies \( \omega_{ij}(e_b) = 0 \). As a result, \( e_1 \) is a parallel vector field on \( M \). Thus \( M \) is flat. Q.E.D.

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**REFERENCES**
