A CHARACTERIZATION OF THE UNIFORM CLOSURE OF
THE SET OF HOMEOMORPHISMS OF
A COMPACT TOTALLY DISCONNECTED METRIC SPACE
INTO ITSELF

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ABSTRACT. The limit index $\lambda(x)$ of a point $x$ in a compact metric space is defined. (Roughly: Isolated points have index 0, limit points have index 1, limit points of limit points have index 2, and so forth.) Then the following theorem is proved.

Theorem 1. Let $E$ be a compact, totally disconnected metric space. Then the uniform closure of the set of homeomorphisms of $E$ into itself is the set $C^\lambda$ of continuous functions $f$ from $E$ to $E$ satisfying

1. $\lambda(x) < \lambda(f(x))$ for all $x \in E$, and
2. if $y$ is not a condensation point of $E$, then $f^{-1}(y)$ contains at most one $x$ such that $\lambda(x) = \lambda(y)$.

Further, the set of homeomorphisms of $E$ into $E$ is a dense $G_\delta$ subset of the complete metric space $C^\lambda$.

A concept that we will call the limit index of a point in a compact metric space was used by Miles in the proof of a theorem in abstract harmonic analysis [1, Theorem A]. Theorem 1 of this paper can be proved from that theorem. The proof of Theorem 1 presented in this paper is simpler but similar and does not use harmonic analysis. The original form of the category argument used here is due to Kaufman [2]. Adaptations have appeared in [1, 3 and 4].

We first introduce some definitions and notation.

Let $E$ be a compact metric space. For each ordinal $\alpha < \Omega$ (the first uncountable ordinal), define $E_\alpha$ as follows. Let $E_0 = E$. Let $E_{\alpha+1}$ be the set of limit points of $E_\alpha$. If $\beta$ is a limit ordinal, let $E_\beta = \cap_{\alpha < \beta} E_\alpha$. (These definitions are due originally to Cantor [5]. See also Kuratowski [6, p. 261].)

It is shown in [1] and in [6, p. 262] that $E_\alpha = E_{\alpha+1}$ for some $\alpha < \Omega$. Let $E_\alpha$ be the first ordinal for which this holds and write $\tilde{E}$ for $E_{\alpha_\varepsilon}$. Observe that $\tilde{E}$ is the set of condensation points of $E$.

For a nonempty closed subset $F$ of $E$, define the limit index of $F$, denoted $\lambda(F)$, as follows: If $F \cap \tilde{E} \neq \emptyset$, let $\lambda(F) = \alpha_\varepsilon$; otherwise let $\lambda(F)$ be the last $\alpha$ such that $F \cap E_\alpha \neq \emptyset$. (A compactness argument, given in [1], shows that such an $\alpha$ exists.) For $x \in E$, we write $\lambda(x)$ for $\lambda(\{x\})$. 
Observe that $\lambda$ has the following properties:

(i) If $\alpha < \alpha_E$, then $\lambda(x) > \alpha$ if and only if $x \in E_\alpha$.

(ii) $\lambda(F) < \alpha_E$ implies that $F \cap E_{\lambda(F)}$ is finite.

(iii) $y \in F$ implies that $\lambda(y) < \lambda(F)$.

Let $C(E, E)$ be the set of continuous functions from $E$ to $E$ and $C(E, R)$ be the set of continuous real-valued functions on $E$. Let $C_{\text{fin}}$ be the set of continuous real-valued functions on $E$ with finite range. For $h \in C(E, R)$ and $\varepsilon > 0$, let $G(h, \varepsilon) = \{ f \in C_\lambda : \| \gamma \circ f - h \|_\infty < \varepsilon \text{ for some } \gamma \in C_{\text{fin}} \}$.

Let $d$ be a metric on $E$ compatible with the topology of $E$. For $f$ and $g$ in $C(E, E)$, let $D(f, g) = \sup\{ d(f(x), g(x)) : x \in E \}$.

**Theorem 2.** Every homeomorphism of $E$ into itself is an element of $C_\lambda$.

**Proof.** Let $f$ be a homeomorphism of $E$ into $E$. The second condition in the definition of $C_\lambda$ is trivially satisfied, since $f$ is one-to-one. It remains to show that the first condition holds or, equivalently, that $f(E_\alpha) \subseteq E_\alpha$ for all $\alpha$. Assume that $f(E_\alpha) \not\subseteq E_\alpha$ is false for some $\alpha$ and let $\beta$ be the first ordinal for which this happens. We will show that this leads to a contradiction. We have $f(E_\beta) \not\subseteq E_\beta$, but, for $\alpha < \beta$, $f(E_\alpha) \subseteq E_\alpha$. Thus, there is an $x \in E_\beta$ such that $y = f(x) \notin E_\beta$. Let $\lambda(y) = \alpha$. Then $\alpha < \beta$. Consider $g = f|_{E_\alpha}$. Clearly, $g$ is a homeomorphism of $E_\alpha$ into $E_\alpha$. Since $y$ is an isolated point of $E_\alpha$, $g^{-1}(y) = x$ is an isolated point of $E_\alpha$. But $x \in E_\beta$ and is therefore a limit point of $E_\alpha$, so we have a contradiction.

**Theorem 3.** $C_\lambda$ is complete in the topology of uniform convergence.

**Proof.** See [1].

**Lemma 1.** Let $x_1, \ldots, x_n$ be distinct elements of $E$; let $g \in C_\lambda$ and let $\eta > 0$. Then there are distinct elements $y_1, \ldots, y_n$ of $E$ such that $\lambda(x_j) < \lambda(y_j)$ and $d(y_j, g(x_j)) < \eta$ for $1 \leq j \leq n$.

**Proof.** See [1].

**Lemma 2.** Each $G(h, \varepsilon)$ is dense in $C_\lambda$.

**Proof.** Fix $h \in C(E, R)$ and $\varepsilon > 0$. Let $g \in C_\lambda$ and $\eta > 0$. We will show that there is an $f \in G(h, \varepsilon)$ such that $D(f, g) < \eta$.

Write $E = \bigcup_{j=1}^n F_j$, where the $F_j$ are pairwise disjoint, nonvoid, open and closed subsets of $E$, and where $h$ varies less than $\varepsilon$ and $g$ varies less than $\eta/2$ on each $F_j$. Let $\lambda(F_j) = \alpha_j$. If $\alpha_j < \alpha_E$, then $F_j \cap E_{\alpha_j}$ is finite, so that we may suppose without loss of generality that $F_j \cap E_{\alpha_j}$ consists of a single point $x_j$. If $\alpha_j = \alpha_E$, let $x_j$ be any point of $F_j \cap E_{\alpha_j}$. By Lemma 1, there are distinct $y_1, \ldots, y_n$ such that $\lambda(y_j) > \lambda(x_j)$ and $d(y_j, g(x_j)) < \eta/2$, $1 \leq j \leq n$. Define $f(x) = y_j$ when $x \in F_j$. Then $f \in C_\lambda$ and $D(f, g) < \eta$. Now write $E = \bigcup_{j=1}^n A_j$, where the $A_j$ are disjoint open and closed sets and $y_j \in A_j$, $1 \leq j \leq n$. Define $\gamma \in C_{\text{fin}}$ by $\gamma(y) = h(x_j)$ when $y \in A_j$. Then, when $x \in F_j$, we have $|\gamma \circ f(x) - h(x)| = |h(x_j) - h(x)| < \varepsilon$, so $\| \gamma \circ f - h \|_\infty < \varepsilon$.

**Lemma 3.** Each $G(h, \varepsilon)$ is open in $C_\lambda$. 
Proof. Fix \( h \in C(E, R) \) and \( \varepsilon > 0 \). Let \( g \in G(h, \varepsilon) \) and let \( \gamma \in C_{\text{fin}} \) be such that \( \|\gamma \circ g - h\|_{\infty} < \varepsilon \). Let the range of \( \gamma \) be \( \{y_1, \ldots, y_n\} \) and let \( F_j = \gamma^{-1}(y_j), 1 \leq j \leq n \). Let \( \eta > 0 \) be such that \( \eta < \min_{i \neq j} \{\text{dist}(F_i, F_j)\} \). Then if \( f \in C_{\lambda} \) and \( D(f, g) < \eta \) we have for all \( x \) that \( f(x) \in F_j \) if and only if \( g(x) \in F_j \), and, hence, \( \gamma \circ f = \gamma \circ g \), so \( \|\gamma \circ f - h\|_{\infty} < \varepsilon \).

Proof of Theorem 1. Let \( f \in C_{\lambda} \). Then \( f \) is a homeomorphism of \( E \) into \( E \) if and only if \( f \) is one-to-one. Also, if \( f \) is not one-to-one, it is clear that there are an \( h \in C(E, R) \) and \( \varepsilon > 0 \) such that \( f \notin G(h, \varepsilon) \). It follows that \( f \) is a homeomorphism of \( E \) into \( E \) if and only if \( f \) is in every \( G(h, \varepsilon) \).

Let \( \{h_n\}_{n=1}^{\infty} \) be dense in \( C(E, R) \). Then \( f \) is a homeomorphism of \( E \) into \( E \) if and only if \( f \) is in \( \bigcap_{n, k=1}^{\infty} G(h_n, k^{-1}) \). Combining this with Theorem 3 and Lemmas 2 and 3 and applying the Baire Category Theorem, we see that the homeomorphisms in \( C_{\lambda} \) form a dense \( G_\delta \) subset of the complete metric space \( C_{\lambda} \). This, together with Theorem 2, completes the proof.

References


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