GENERATORS OF $H^*(MSO; \mathbb{Z}_2)$ AS A MODULE
OVER THE STEENROD ALGEBRA,
AND THE ORIENTED COBORDISM RING

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Abstract. In this paper we will describe a minimal set of $A$-generators of $H^*(MSO; \mathbb{Z}_2)$ (where $A$ is the mod-2 Steenrod Algebra). The description is very much analogous to R. Thom's description of generators for $H^*(MO; \mathbb{Z}_2)$ (see [7]). As a corollary, we give simple cohomological criteria for a manifold to be indecomposable in the oriented cobordism. Our proof relies on work of D. J. Pengelley (see [5]).\footnote{The original proof was lengthier and elementary. After the original version of the paper was written, I was informed of Pengelley's unpublished work, which could be used to shorten the argument.}

0. Statement of results. In order to describe our results, we need some terminology.

All homology and cohomology groups of this paper will have $\mathbb{Z}_2$ coefficients. $\Omega$ will be the oriented cobordism ring.

The cohomology of $BO$ will be identified, in the well-known way, with the subalgebra of $\mathbb{Z}_2[t_1, t_2, \ldots, t_N]$, which consists of all symmetric polynomials (each time the index $N$ will be big enough for our purposes).

1. Definition. We will call a partition a finite sequence of positive integers $\omega = (a_1, a_2, \ldots, a_k)$, so that $a_1 < a_2 < \cdots < a_k$. We will call the degree of $\omega$ the integer $|\omega| = a_1 + a_2 + \cdots + a_k$. We call the length of $\omega$ the number of terms which appear in $\omega$, i.e. $l(\omega) = k$.

If $\omega$ is a partition, then $s(\omega)$ is the well-known element of $H^{|\omega|}(BO)$, i.e.

$$s(\omega) = \sum t_1^{a_1} t_2^{a_2} \cdots t_k^{a_k}.$$

It is well known that the $s(\omega)$'s form a $\mathbb{Z}_2$-basis for $H^*(BO)$, and the elements of the form $s(\omega) \cdot U$ (where $U \in H^0(MO)$ is the Thom class) constitute a $\mathbb{Z}_2$-basis for $H^*(MO)$ (see [2]).

If $M$ is any closed, compact and $C^\infty$ manifold, then $s(\omega)(M) \in \mathbb{Z}_2$ is the corresponding normal characteristic number of $M$.

Let $I: MSO \to MO$ be the obvious map.

2. Definition. We define $P$ to be the set of all partitions $\omega$ which satisfy all the following conditions.

(a) No number of the form $2^i - 1$, where $i > 1$, is included in $\omega$. 

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(b) A number of the form $2^i$, where $i > 1$, appears always an even number of times in the partition $\omega$. (Remark. The number zero is even.)

We define $P_1$ to be the subset of $P$ which consists of all partitions of the form $(2a_1, 2a_1, 2a_2, 2a_2, \ldots, 2a_k, 2a_k)$ where $0 < a_1 < a_2 < \cdots < a_k$.

We define $P_2$ to be the subset of $(P - P_1)$ which consists of all partitions of the form $(a_1, a_2, \ldots, a_k)$ or of the form $(a_1, a_2, \ldots, a_k, 2b_1, 2b_1, \ldots, 2b_m, 2b_m)$, where $a_k$ is an odd number.

Our main result is the following

3. **Theorem.** The set of elements $\{I^*(s(\omega) \cdot U): \omega \in P_1 \cup P_2\}$ is a minimal set of generators for the $A$-module $H^*(MSO)$. The only relations are

$$\text{Sq}^1(I^*(s(\omega) \cdot U)) = 0, \quad \text{where } \omega \in P_1.$$ 

4. **Definition.** Let $P_3$ be the set of all partitions $(2a, 2a)$, where $a > 0$. Let $P_4$ be the subset of $P_2$ which consists of all partitions of the form $(a_1, a_2, \ldots, a_k)$ where $a_k$ is an odd number and the $a_1, a_2, \ldots, a_{k-1}$'s are unequal even numbers.

The following two theorems are corollaries of Theorem 3.

5. **Theorem.** Let $M$ be an oriented manifold whose oriented cobordism class belongs to the torsion part of $\Omega$. The manifold $M$ is indecomposable in $\Omega$ if and only if there is a partition $\omega \in P_4$ so that $s(\omega)(M) \neq 0$.

The corresponding condition for the free part of $\Omega$ is well known (see [8, p. 293]).

6. **Theorem.** Let $\{M_{4k}: k > 1\}$ be a collection of oriented manifolds which form a minimal set of generators of the free part of $\Omega$. Let $\{M_\omega: \omega \in P_4\}$ be a collection of oriented manifolds so that $\dim M_\omega = |\omega|$. Then, the collection of manifolds

$$\{M_{4k}, M_\omega: k > 1, \omega \in P_4\},$$

is a minimal set of generators for $\Omega$ if and only if the matrix $||s(\omega')(M_\omega)||$, where $\omega, \omega' \in P_4$, is invertible.

1. **The $A_\omega$-comodule structure of $H_\omega(MSO)$.** The main result of this section is Theorem 12, which is a corollary of D. Pengelley’s work (see Theorem 8) and provides certain information concerning the $A_\omega$-comodule structure of $H_\omega(MSO)$.

(Remark. $A_\omega$ is the dual of the mod-2 Steenrod Algebra.)

Let $\{x(\omega): \omega$ is a partition$\}$ be the basis of $H_\omega(MO)$, which is dual to the basis $(s(\omega) \cdot U: \omega$ is a partition$)$ of $H^*(MO)$, and let $x(\omega)$ be the dual of $s(\omega) \cdot U$.

The following theorem is well known.

7. **Theorem.** Let $x_i = x((i))$, where $i > 0$, and let $\omega = (a_1, a_2, \ldots, a_k)$ be a partition. Then $H_\omega(MO)$ is a polynomial algebra so that

$$H_\omega(MO) = \mathbb{Z}_2[x_1, x_2, \ldots, x_n, \ldots].$$

Furthermore, we have $x(\omega) = x_{a_1}x_{a_2} \cdots x_{a_k}$.

**Proof.** See, for example, [1].

Let $\tilde{\xi}_i \in A_\omega(2^i-1)$ be the Hopf Algebra conjugate of Milnor’s generators $\xi_i \in A_\omega(2^i-1)$.
8. Theorem (D. J. Pengelley). There is a sequence of elements $y_n \in H_n(MO)$, where $n \geq 2$, so that

$$I_\bullet(H_\bullet(MO)) = \mathbb{Z}_2[y_2, y_3, \ldots, y_n, \ldots].$$

If $n \neq 2^i$, then $y_n$ is indecomposable in $H_\bullet(MO)$. If $n = 2^i$, where $i > 1$, then there is an indecomposable element $z_{n/2} \in H_{n/2}(MO)$, so that $y_n = (z_{n/2})^2$. Furthermore, if $\mu_\bullet : H_\bullet(MO) \to A_\bullet \otimes H_\bullet(MO)$ is the obvious coaction map, then we have

$$\mu_\bullet(y_n) = \begin{cases} 
\tilde{\xi}_1 \otimes 1 + 1 \otimes y_2, & \text{if } n = 2, \\
1 \otimes (z_{n/2})^2, & \text{if } n = 2^i \text{ and } i > 2, \\
\sum_{j=0}^{i} \tilde{\xi}_j \otimes y_{2^i-j-1}, & \text{if } n = 2^i - 1 \text{ and } i > 2, \\
1 \otimes y_n + \tilde{\xi}_1 \otimes y_{n-1}, & \text{if } n = 2k \text{ and } k \neq 2^i, \\
1 \otimes y_n, & \text{otherwise.}
\end{cases}$$

Proof. See [5].

9. Definition. Let $\omega_1$, $\omega_2$ be two partitions. We say that $\omega_1$ is bigger than $\omega_2$ if and only if at least one of the following two conditions is satisfied:

(a) $l(\omega_1) > l(\omega_2)$.
(b) $l(\omega_1) = l(\omega_2)$ and $|\omega_1| < |\omega_2|$.

This relation of “bigger” is clearly transitive but it is not a total ordering.

10. Definition. Let $\omega_1$, $\omega_2$, $\ldots$, $\omega_k$ be $k$ distinct partitions and let $\omega$ be another partition. Let $a$, $a_1$, $\ldots$, $a_k$ be nonzero elements of $A_\bullet$. We say that the element

$$a_1 \otimes x(\omega_1) + a_2 \otimes x(\omega_2) + \cdots + a_k \otimes x(\omega_k)$$

of $A_\bullet \otimes H_\bullet(MO)$ is bigger than $a \otimes x(\omega)$ if and only if all the partitions $\omega_1$, $\omega_2$, $\ldots$, $\omega_k$ are bigger than $\omega$.

11. Definition. Let $x, y \in A_\bullet \otimes H_\bullet(MO)$. We define the symbol $x < y$ to mean that the element $(x - y)$ is bigger than $x$, or that $(x - y) = 0$.

Remark. We caution the reader about the fact that the relation $<$ is not defined for arbitrary elements of $A_\bullet \otimes H_\bullet(MO)$.

Our next result is a corollary of Theorem 8.

12. Theorem. We have

(a) $\tilde{\xi}_1 \otimes 1 < \mu_\bullet(y_2)$.

(b) If $n = 2^i$ and $i > 2$, then

$$1 \otimes x_{(n/2)}^2 < \mu_\bullet(y_n).$$

(c) If $n = 2^i - 1$ and $i > 2$, then

$$\tilde{\xi}_i \otimes 1 < \mu_\bullet(y_n).$$

(d) If $n \neq 2^i$, $2^i - 1$, for $i > 0$, then

$$i \otimes x_n < \mu_\bullet(y_n).$$
2. The Steenrod Algebra. In this section we will describe two well-known lemmas about the Steenrod Algebra, which will be used in the sequel.

13. Lemma. Let $B$ be the subspace of $A$ generated by the elements $Sq^1Sq^2 \ldots Sq^k$ where $i_{t-1} > 2i_t$, $k > t > 2$ and $i_k > 2$. Then $A$ is the direct sum of the subspaces $B$ and $B \cdot Sq^1$. Furthermore $A \cdot Sq^1 = B \cdot Sq^1$.

Proof. See [4, p. 7–8].

14. Lemma. The subspace of $A_*$ which is the annihilator of $A \cdot Sq^1$ is the polynomial subalgebra of $A_*$ generated by $\xi_1^2, \xi_i$ for $i > 2$.

Proof. Let $Sq^R$, where $R = (r_1, r_2, \ldots)$, be the well-known element of the Milnor $s$ basis of $A$ (see [3]). Milnor proves that

$$Sq^1Sq^R = (r_1 + 1)Sq^{r_1 + 1,r_2 \ldots}.$$ 

This implies that the elements $\xi_1^2, \xi_i$ for $i > 2$, belong to the annihilator of $A \cdot Sq^1$. The rest of the proof follows from the dimensions of the $\mathbb{Z}_2$-spaces $B$, $B \cdot Sq^1$, $\mathbb{Z}_2[\xi_1, \xi_2, \xi_3, \ldots]$.

3. Proof of Theorem 3. In this section we will prove Theorem 3, but first we need some preparation.

15. Definition. Let $X$ be a subset of a vector space. Then $\langle X \rangle$ is the subspace spanned by $X$.

Let $C$ be a set of partitions. Then we define $s(C) = \{ s(\omega) : \omega \in C \}$.
Let $\omega = (a_1, a_2, \ldots, a_k)$ be a partition. Then we define $y(\omega) = y_{a_1}y_{a_2} \cdots y_{a_k}$.

16. Proposition. The restriction of the map $I^* \mu$,

$$I^* \mu : B \otimes \langle s(P) \cdot U \rangle \to H^*(MSO)$$

is an isomorphism.

(Remark. For the definition of $B$, see Lemma 13. For the definition of $P$, see Definition 2.)

Proof. First we observe that the graded spaces $B \otimes \langle s(P) \cdot U \rangle$ and $H^*(MSO)$ have the same $\mathbb{Z}_2$-dimensions in each degree. This follows easily from the definition of $B$, $P$ and the cohomology of $MSO$.

So it is enough to prove that the restriction of the map $I^* \mu$ is a monomorphism. We argue by contradiction. Let us assume that there are $k$ distinct partitions $\omega_1, \omega_2, \ldots, \omega_k$ of $P$ and nonzero elements of $B, a_1, a_2, \ldots, a_k$, so that

$$I^* \mu(a_1 \otimes s(\omega_1)U + a_2 \otimes s(\omega_2)U + \cdots + a_k \otimes s(\omega_k)U) = 0.$$ 

Among the partitions $\omega_1, \omega_2, \ldots, \omega_k$, there is at least one which is maximal in the relation bigger. Let us suppose that $\omega_1$ is such a maximal partition. Let $\omega$ be the partition that we get by substituting in $\omega_1$ every occurrence of $(\ldots, 2^i, 2^i, \ldots)$ by $(\ldots, 2^{i+1}, \ldots)$. Then, by Theorem 12, we have $1 \otimes x(\omega_1) < \mu_a(y(\omega))$. Besides, there is an element $b$ belonging to the annihilator of $A \cdot Sq^1 = B \cdot Sq^1$, so that $\langle a_1, b \rangle = 1$. Furthermore, by Lemma 14, the element $b$ can be chosen to be a
product of $\xi_i^2$, $\xi_i$'s where $i > 2$. So, by Theorem 12, there is a partition $\omega'$, consisting entirely of 2, $(2^i - 1)$'s where $i > 2$, so that

$$b \otimes 1 < \mu_*(y(\omega')).$$

Finally, by Theorem 8, there is an element $z \in H_*(MSO)$ so that $I_*(z) = y(\omega')y(\omega)$.

Combining all the above we get $I_*(z) = y(\omega')y(\omega_1)$. Combining the above, we have

$$< I^*(a_1 \otimes s(\omega_1) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U), z >$$

$$= < a_1 \otimes s(\omega) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U, \mu_*(I_*(z)) >$$

$$= < a_1 \otimes s(\omega), \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U, \mu_*(y(\omega)y(\omega)) >$$

$$= < a_1 \otimes s(\omega_1) \cdot U + \cdots + a_k \otimes s(\omega_k) \cdot U, b \otimes x(\omega_1) >$$

$$= < a_1 \otimes s(\omega_1) \cdot U, b \otimes x(\omega_1) > = 1 \neq 0$$

which contradicts our assumption.

17. **Lemma.** Let $\omega \in (P - (P_1 \cup P_2))$ be a partition. Then there is a partition $\omega_0 \in P_2$ so that $s(\omega) - Sq^1s(\omega_0) = \Sigma s(\omega_i)$ where the $\omega_i$'s belong to $P_2$.

**Proof.** Let $\omega = (a_1, \ldots, a_m, b_1, b_1, \ldots, b_k, b_k)$ where $a_m$ is an even positive integer and $a_{m-1} < a_m$. Then we define $\omega_0 = (a_1, \ldots, a_{m-1}, a_m - 1, b_1, b_1, \ldots, b_k, b_k)$. Clearly $\omega_0 \in P_2$. The assertion of the lemma follows easily.

18. **Proposition.** Let $R$ be the subalgebra of $A$ generated by the element $Sq^1$. Then the $Z_2$-space $I^*(s(P - P_1) \cdot U)$ is a free $R$-module and the set $I^*(s(P_2) \cdot U)$ is a free basis.

**Proof.** The previous lemma says that the set $I^*(s(P_2) \cdot U)$ is a set of $R$-generators for the $R$-module $I^*(s(P - P_1) \cdot U)$. (Remark. Note that $Sq^1I^*(s(\omega) \cdot U) = I^*(Sq^1(s(\omega)) \cdot U)$). Next, it is not difficult to observe that the number of partitions of $(P - P_1 \cup P_2)$ of degree $m$ equals the number of partitions of $P_2$ of degree $(m - 1)$. This implies that the set of elements $\{s(\omega) \cdot U, Sq^1(s(\omega) \cdot U)\}$ for $\omega \in P_2$ is $Z_2$-independent. That ends the proof.

**Proof of Theorem 3.** It follows easily from the results of this section.

4. **Proof of Theorems 5 and 6.** In this final section, we will complete the proofs of Theorems 5 and 6, but first we will need some preparation.

19. **Lemma.** Let $\omega$ be a partition consisting entirely of even numbers, so that at least one of them appears an odd number of times in $\omega$. If $M$ is an orientable manifold, then $s(\omega)(M) = 0$.

**Proof.** Let $\omega = (a_1, a_2, \ldots, a_m)$, so that $a_{m-1} < a_m$ and the number $a_m$ appears in $\omega$ an odd number of times. Let $\omega_0 = (a_1, a_2, \ldots, a_m - 1, \ldots, a_k)$.
Clearly
\[ \text{Sq}^1(I^*(s(\omega_0) \cdot U)) = I^*(s(\omega) \cdot U). \]
Now the proof follows without charge.

20. Corollary. Let \( \omega \in P_3 \cup P_4 \) and let \( M \) be an oriented manifold which is decomposable in \( \Omega \). Then
\[ s(\omega)(M) = 0. \]

21. Definition. Let \( \omega \in P_1 \cup P_2 \). Then \( N_\omega \) is defined to be an oriented manifold, so that \( s(\omega)(N_\omega) \neq 0 \) and \( s(\omega')(N_\omega) = 0 \) for all \( \omega' \in P_1 \cup P_2 \) and \( \omega' \neq \omega \). The existence of such manifolds is guaranteed by Theorem 3.

22. Proposition. The family of \( N_\omega \)'s, where \( \omega \in P_3 \cup P_4 \), is a minimal set of algebra generators for \( \Omega \otimes \mathbb{Z}_2 \).

Proof. By the previous corollary, the cobordism classes of \( N_\omega \)'s are linearly independent in \( (\Omega \otimes \mathbb{Z}_2)/(\text{decomposable}) \). On the other hand, these manifolds are as numerous as Wall's generators of \( \Omega \) (see [8, p. 309]). So, they must generate \( \Omega \otimes \mathbb{Z}_2 \).

23. Corollary. Let \( \omega \in P_3 \). Then the manifold \( N_\omega \) of Definition 21 can be chosen to be a polynomial generator of the torsion free part of \( \Omega \).

Now the proof of Theorems 5 and 6 follows without difficulty from Proposition 22 and Corollary 23.

**Bibliography**


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