A NOTE ON IDEALS IN THE DISC ALGEBRA

FRANK FORELLI

ABSTRACT. We offer an elementary theorem on ideals in the disc algebra \( A(\overline{D}) \), which by way of a corollary, one, identifies the maximal ideals of \( A(\overline{D}) \), and two, provides a proof, which avoids the axiom of choice, that every proper ideal in \( A(\overline{D}) \) is contained in a maximal ideal.

1. Let \( A \) be a ring of functions in the closed unit disc \( \overline{D} \), each of which is continuous in \( \overline{D} \) and holomorphic in \( D \). (Thus \( A \subseteq A(\overline{D}) \).) Suppose that
(a) \( A \) is dense in the disc algebra \( A(\overline{D}) \), i.e. if \( f \in A(\overline{D}) \) and \( \varepsilon > 0 \), then there is a \( g \) in \( A \) with \( |f - g| < \varepsilon \) in \( D \).

Then we have the following.

THEOREM. Let \( Q \) be an ideal in \( A \), \( Q \neq 0 \), and let

\[ X \subset \bigcup_{f \in Q} \{ f \neq 0 \} \]

where by \( \{ f \neq 0 \} \) we mean the set of those \( z \) in \( \overline{D} \) for which \( f(z) \neq 0 \). If \( X \) is closed in \( \overline{D} \), or if \( A = A(\overline{D}) \) and

\[ X \cap T = \overline{X} \cap T \]

where by \( \overline{X} \) we mean the closure of \( X \) in \( \overline{D} \), then there is an \( f \) in \( Q \) with \( f \) vanishing nowhere in \( X \).

We will come to the proof in due course. The theorem raises the following question (that we are unable to answer). Which ideals \( Q \) in \( A \), \( Q \neq 0 \), have the property that if \( X \) is equal to the right side of (1), then there is an \( f \) in \( Q \) with \( f \) vanishing nowhere in \( X \)? This holds if \( Q \) is principal. What if \( Q \) is finitely generated?

Does it hold if \( Q \) is an ideal of denominators, i.e. if

\[ Q = Q(\gamma) = \{ f : f \in A, f\gamma \in A \} \]

where \( \gamma \) is in the field of fractions \( A(0) \) of \( A \)? If yes, this would imply (see the proof of the corollary below) that if \( X \) is any subset of \( \overline{D}, X \neq \emptyset \), then

\[ A_X = \bigcap_{\gamma \in X} A_{\gamma} \]

where by \( A_X \) we mean the ring of fractions

\[ \{ g/f : g, f \in A, f \text{ vanishes nowhere in } X \} \].

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and by $A_\zeta$, $\zeta \in \overline{D}$, we mean the local ring of fractions
\[ \{ g/f: g, f \in A, f(\zeta) \neq 0 \}. \]
If not, then one might ask for which $X \subset \overline{D}$ does (4) hold. In this regard we have the following.

**Corollary.** Let $X \subset \overline{D}$, $X \neq \emptyset$. If $X$ is closed in $\overline{D}$, or if $A = A(D)$ and (2) holds, then (4) holds.

**Proof.** Let $\gamma$ belong to the right side of (4), and put $Q =$ the right side of (3). Then (1) holds; hence there is an $f$ in $Q$ with $f$ vanishing nowhere in $X$. This proves that $\gamma \in A_X$.

Suppose that in addition to (a) we have

(b) If $f \in A$, and $f$ vanishes nowhere in $\overline{D}$, then $1/f \in A$.

Then the following holds.

**Corollary.** Let $Q$ be an ideal in $A$, $Q \neq A$. Then there is a $\zeta$ in $\overline{D}$ such that $Q \subset P_\zeta$ where
\[ P_\zeta = \{ f: f \in A, f(\zeta) = 0 \}. \]

**Proof.** Otherwise, in the statement of the theorem we may take $X = \overline{D}$.

Our proof of the theorem is very elementary; its main ingredient is that if
\[ \varphi = \sum_{-k}^{k} \alpha_j e^{ij\theta} \]
is a trigonometric polynomial, then
\[ e^{ik\theta} \varphi = g | T \]
where $g \in A(D)$.

By the second corollary, every maximal ideal in $A$ is a $P_\zeta$, $\zeta \in \overline{D}$; on the other hand, if $A$ is an algebra over $\mathbb{C}$, then every $P_\zeta$ is maximal. This identifies the maximal ideals of $A(D)$ in a way which is more elementary than those in [3]. To turn the second corollary around, we have an easy and elementary proof (which does not use the axiom of choice) of the fact that if $f_1, \ldots, f_n$ in $A$ do not have a common zero in $\overline{D}$, then $(f_1, \ldots, f_n) = (1)$. Other proofs of this (if $A = A(D)$) which avoid the axiom of choice are in [1] and [2].

We might point out that Theorem 1 is to some extent peculiar to the disc $\mathbb{D}$. For example, it fails for the ball algebra $A(B)$ in 2 variables (let $Q = P_0 = \{ f: f \in A(B), f(0) = 0 \}$, $X = \partial B = \{ (z, w): |z|^2 + |w|^2 = 1 \}$).

**2.** We now come to the proof of the theorem. To begin, let $A = A(D)$. Since $X \cap T$ is compact, we have
\[ X \cap T \subset \bigcup_{j=1}^{n} \{ f_j \neq 0 \} \]
where \( f_1, \ldots, f_n \in \mathcal{Q} \). Let \( Y = X \cap T \). WLOG we may assume that \( Y \neq \emptyset \). Put

\[
f = \sum_{j=1}^{n} f_j f_j,
\]

let

\[
\alpha = \inf_{Y} f, \quad \beta = \inf_{T} \left( \frac{1}{\sum_{j=1}^{n} |f_j|} \right),
\]

and choose trigonometric polynomials \( \varphi_1, \ldots, \varphi_n \) such that \( |f_j - \varphi_j| < \alpha \beta \). Then (on \( T \))

\[
\left| f - \sum_{j=1}^{n} \varphi_j f_j \right| = \left| \sum_{j=1}^{n} (f_j - \varphi_j) f_j \right| < \alpha,
\]

hence \( \sum \varphi_j f_j \) vanishes nowhere in \( Y \). Replacing \( \varphi_j \) by \( e^{ik\theta} \varphi_j \) (where \( k \geq \deg \varphi_j, 1 \leq j \leq n \)), we obtain (cf. (5)) \( g_1, \ldots, g_n \) in \( A(D) \) with \( \sum g_j f_j \) vanishing nowhere in \( Y \).

Let

\[
g = \sum_{j=1}^{n} g_j f_j.
\]

Then \( g \in (f_1, \ldots, f_n) \). Since \( g \) vanishes nowhere in \( X \cap T \), \( g \) has at most a finite number of zeros in \( X \) (counting multiplicities), say \( \xi_1, \ldots, \xi_m \). We have \( \xi_j \in D \), hence

\[
g(z) = (z - \xi_1) \cdots (z - \xi_m) h(z)
\]

where \( h \in A(D), h \) vanishes nowhere in \( X \).

Let \( 1 \leq j \leq m \), and choose \( h_j \in \mathcal{Q} \) with \( h_j(\xi_j) \neq 0 \). We have

\[
h_j(z) - h_j(\xi_j) = (z - \xi_j) \psi_j(z)
\]

where \( \psi_j \in A(D), \) i.e.

\[
h_j(\xi_j) = h_j - (z - \xi_j) \psi_j.
\]

Put

\[
\mu = \prod_{j=1}^{m} h_j(\xi_j);
\]

then (6) and (7) give

\[
\mu h \in (f_1, \ldots, f_n, h_1, \ldots, h_m).
\]

Thus \( h \in \mathcal{Q} \), which completes the proof of the theorem if \( A = A(D) \).

If \( A \neq A(D) \), put

\[
P = \left\{ \sum_{1}^{k} f_j g_j : f_j \in A(D), g_j \in \mathcal{Q} \right\}.
\]

Then \( P \) is an ideal in \( A(D) \), with \( Q \subset P \); by the foregoing there is an \( f \) in \( P \) with \( f \) vanishing nowhere in \( X \). We have

\[
f = \sum_{j=1}^{n} f_j g_j
\]

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where \( f_j \in A(\mathbb{D}), g_j \in Q \). Let

\[
\alpha = \inf_{x} |f|, \quad \beta = \inf_{\mathbb{D}} \left( \frac{1}{\sum_{j=1}^{n} |g_j|} \right).
\]

By the hypothesis (a) we may choose \( h_j \) in \( A, 1 \leq j \leq n \), with \( |f_j - h_j| < \alpha \beta \). Put

\[
h = \sum_{j=1}^{n} h_j g_j.
\]

Then \( h \in Q \); and

\[
|f - h| = \left| \sum_{j=1}^{n} (f_j - h_j) g_j \right| < \alpha,
\]

hence \( h \) vanishes nowhere in \( X \).

3. In the foregoing we may replace \( \mathbb{D} \) by any bordered Riemann surface \( \bar{W} \) which is compact. Our proof of the theorem (and its corollaries) would then work for any ring of functions in \( A(\bar{W}) \) which is dense in \( A(W) \). One should replace the trigonometric polynomials on \( T \) by the separating selfadjoint algebra in \( C(\partial W) \) consisting of the complex linear span of quotients of inner functions in \( A(W) \) (cf. [4], where it is proved that the inner functions in \( A(W) \) separate points in \( \partial W \)).

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

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