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SPHERICAL HARMONICS GENERATING BOUNDED BIHARMONICS

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ABSTRACT. Let $H^2B(R)$ be the family of bounded nonharmonic biharmonic functions on a Riemannian manifold R. On the punctured Euclidean N-space $E_0^N = \{x = (x^1,\ldots,x^N) \mid 0 < |x| < \infty\}$, H^2B is void for $N > 3$, whereas for $N = 2, 3$, it is generated by certain fundamental spherical harmonics. It is also known that H^2B remains void on the Riemannian manifold E_α^N, $N > 3$, obtained by endowing E_0^N with the non-Euclidean metric $ds_\alpha = r^\alpha |dx|$, $\alpha \in R$.

The purpose of the present paper is to show that the fundamental spherical harmonics continue generating $H^2B(E_\alpha^3)$, despite the distorting metric ds_α. An analogous result holds for E_α^2.

Let $H^2B(R)$ be the family of bounded nonharmonic biharmonic functions on a Riemannian manifold R. It was shown in Sario and Wang [2] that on the punctured Euclidean N-space $E_0^N = \{x = (x^1,\ldots,x^N) \mid 0 < |x| < \infty\}$, H^2B is void for $N > 3$, whereas for $N = 2, 3$, it is generated by certain fundamental spherical harmonics.

In Sario and Wang [3], it was proved that H^2B remains void on the Riemannian manifold E_α^N, $N > 3$, obtained by endowing E_0^N with the non-Euclidean metric $ds_\alpha = r^\alpha |dx|$, $\alpha \in \mathbf{R}$. The cases $N = 2, 3$ were left open.

The purpose of the present paper is to show that the fundamental spherical harmonics continue generating $H^2B(E_\alpha^3)$, despite the distorting metric ds_α. An analogous result holds for E_α^2.

1.1. Consider the space

$$E_\alpha^N = \left\{x = \left(r, \theta^1,\ldots,\theta^{N-1}\right) \mid 0 < r < \infty, ds_\alpha = r^\alpha |dx|, \alpha \in \mathbf{R}\right\}$$

with polar coordinates $(r, \theta) = (r, \theta^1,\ldots,\theta^{N-1})$. We first state some facts from [3].

LEMMA 1. *There exist no H^2B functions on E_{-1}^N for $N \geqslant 2$.*

Thus when we are searching for generators of H^2B functions, we may ignore the case of $\alpha = -1$.

LEMMA 2. *For any $\alpha \neq -1$, and $N \geqslant 2$, every $h \in H(E_\alpha^N)$ has an expansion*

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n}) S_{nm} + a\sigma(r) + b,$$

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where

$$\sigma(r) = \begin{cases} \log r, & \text{if } N = 2, \\ r^{-(N-2)(\alpha+1)}, & \text{if } N > 2. \end{cases}$$

LEMMA 3. *Let u be a biharmonic function on E_α^N, $N \geqslant 2$, $\alpha \neq -1$, with*

$$\Delta u = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n}) S_{nm} + a\sigma(r) + b.$$

Then

$$\begin{aligned} u = {} & \frac{-1}{4(\alpha+1)} \left[\sum_{n \neq \mu} \sum_{m=1}^{m_n} \frac{1}{P_n} a_{nm} r^{p_n + 2\alpha + 2} S_{nm} + \sum_{n \neq \nu} \sum_{m=1}^{m_n} \frac{1}{Q_n} b_{nm} r^{q_n + 2\alpha + 2} S_{nm} \right] \\ & - \frac{1}{2(\alpha+1)} r^{(2-N/2)(\alpha+1)} \log r \cdot \left(\sum_{m=1}^{m_\mu} a_{\mu m} S_{\mu m} + \sum_{m=1}^{m_\nu} b_{\nu m} S_{\nu m} \right) \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (c_{nm} r^{p_n} + d_{nm} r^{q_n}) S_{nm} + a\tau(r) + bs(r) + c\sigma(r) + d, \end{aligned}$$

where

$$p_n = \tfrac{1}{2}\left\{-(N-2)(\alpha+1) + \left[(N-2)^2(\alpha+1)^2 + 4n(n+N-2)\right]^{1/2}\right\},$$

$$q_n = \tfrac{1}{2}\left\{-(N-2)(\alpha+1) - \left[(N-2)^2(\alpha+1)^2 + 4n(n+N-2)\right]^{1/2}\right\},$$

$$P_n = \tfrac{1}{2}N(\alpha+1) + p_n, \qquad Q_n = \tfrac{1}{2}N(\alpha+1) + q_n,$$

μ and ν are defined by $P_\mu = 0$ and $Q_\nu = 0$ respectively, $\sigma(r)$ is as above,

$$s(r) = \frac{-1}{2N(\alpha+1)^2} r^{2\alpha+2},$$

and

$$\tau(r) = \begin{cases} s(r)\left[\log r - \dfrac{1}{\alpha+1}\right], & N = 2, \\ -\dfrac{1}{2\alpha+2} \log r, & N = 4, \\ \dfrac{1}{2(N-4)(\alpha+1)^2} r^{-(N-4)(\alpha+1)}, & N \neq 2, 4. \end{cases}$$

1.2. Generators of H^2B-functions on E_α^2. *We claim*

THEOREM 1. *$H^2B(E_\alpha^2)$, where $\alpha = -1 \pm k/2$, $k \in Z^+$, is generated by* 1, $\sin k\theta$, *and* $\cos k\theta$.

PROOF. We shall establish the theorem for the case $\alpha = -1 - k/2$, the other case being entirely similar. Let $S_{n1} = \cos n\theta$ and $S_{n2} = \sin n\theta$. From Lemma 3 we calculate

$$\begin{aligned} p_n &= \tfrac{1}{2}\left\{-(N-2)(\alpha+1) + \left[(N-2)^2(\alpha+1)^2 + 4n(n+N-2)\right]^{1/2}\right\} \\ &= \tfrac{1}{2}\left[(4n^2)\right]^{1/2} = n, \\ q_n &= \tfrac{1}{2}\left\{-(N-2)(\alpha+1) - \left[(N-2)^2(\alpha+1)^2 + 4n(n+N-2)\right]^{1/2}\right\} \\ &= -\tfrac{1}{2}\left[(4n^2)\right]^{1/2} = -n. \end{aligned}$$

$P_n = 0$ implies $\frac{1}{2}N(\alpha+1) + p_n = 0$, which yields $n = \frac{k}{2}$. Thus $\mu = \frac{k}{2}$ if k is even. Since μ must be an integer, there is no μ term if k is odd.

$Q_n = 0$ gives $\frac{1}{2}N(\alpha+1) + q_n = 0$, whence $-n = \frac{k}{2}$, which is impossible, since both n and k are positive. Thus there is no ν term.

We may represent u by

$$\begin{aligned} u = &\sum_{n \neq k/2} \sum_{m=1}^{2} A_{nm} r^{n-k} S_{nm} + \sum_{n=1}^{\infty} \sum_{m=1}^{2} B_{nm} r^{-n-k} S_{nm} \\ &+ r^{-k/2} \log r \cdot \sum_{m=1}^{2} a_{(k/2)m} S_{(k/2)m} + \sum_{n=1}^{\infty} \sum_{m=1}^{2} (c_{nm} r^n + d_{nm} r^{-n}) S_{nm} \\ &+ a\tau(r) + bs(r) + c\sigma(r) + d. \end{aligned}$$

We wish to show that all coefficients are zero except possibly A_{k1}, A_{k2} and d.

Suppose $c_{nm} \neq 0$ for some (n, m). Choose $\rho(r) \in C_0^\infty$, $\rho \geqslant 0$, $\operatorname{supp} \rho \subset (0, 1)$. Define $\rho_t(r) = \rho(r - t)$ for $0 < t < \infty$, and $\varphi_t(r, \theta) = \rho_t S_{nm}$. Then $\operatorname{supp} \varphi_t \subset (t, t + 1)$, and by the orthogonality of the $\{S_{nm}\}$,

$$\begin{aligned} (u, \varphi_t) &= \int_{E_\alpha^2} u \rho_t S_{nm} r^{-k-1}\, d\theta\, dr \\ &= c \int_t^{t+1} \Big[A_{nm} r^{n-k} + B_{nm} r^{-n-k} \\ &\qquad + a_{(k/2)m} r^{-k/2} \log r + c_{nm} r^n + d_{nm} r^{-n} \Big] \rho_t r^{-k-1}\, dr, \end{aligned}$$

where the third term in the integrand appears only if $n = k/2$; also,

$$(1, |\varphi_t|) = c \int_t^{t+1} \rho_t r^{-k-1}\, dr,$$

the c's denoting constants, not necessarily the same.

Certainly $\varphi_t \in L^1(E_\alpha^2)$, and thus we have the inequality

$$|(u, \varphi_t)| \leqslant (\sup |u|)(1, |\varphi_t|).$$

But as $t \to \infty$, $(u, \varphi_t) \sim ct^{n-k-1}$ and $(1, |\varphi_t|) \sim ct^{-k-1}$, thus forcing $c_{nm} = 0$ for all n and m.

If $A_{nm} \neq 0$ for $n > k$, the same estimate with the same test function as in the first step gives, as $t \to \infty$,

$$(u, \varphi_t) \sim ct^{n-2k-1}, \qquad (1, |\varphi_t|) = O(t^{-k-1}).$$

Therefore $A_{nm} = 0$ for $n > k$.

Now suppose $B_{nm} \neq 0$ for some (n, m). Choose $\rho(r) \in C_0^\infty$, $\rho \geqslant 0$, $\operatorname{supp} \rho \subset (\beta, \gamma)$, $0 < \beta < \gamma < 1$. Define $\rho_t(r) = \rho(r/t)$ for $0 < t < 1$, and $\varphi_t(r, \theta) = \rho_t S_{nm}$. Then

$$(u, \varphi_t) = c\int_{\beta t}^{\gamma t} \left[A_{nm} r^{n-k} + B_{nm} r^{-n-k} + a_{(k/2)m} r^{-k/2} \log r + d_{nm} r^{-n}\right] \rho_t r^{-k-1}\, dr,$$

where the first term in the integrand appears only if $n < k$, and the third only if $n = k/2$. On the other hand,

$$(1, |\varphi_t|) = c\int_{\beta t}^{\gamma t} \rho_t r^{-k-1}\, dr.$$

As $t \to 0$,

$$(u, \varphi_t) \sim ct^{-n-2k}, \qquad (1, |\varphi_t|) = O(t^{-k}),$$

and again we have a contradiction to $\varphi_t \in L^1(E_\alpha^2)$. Therefore, $B_{nm} = 0$ for all m and n.

Suppose $d_{mn} \neq 0$ for some $n \geqslant k$. Using the same test function as above we obtain

$$(u, \varphi_t) = c\int_{\beta t}^{\gamma t} d_{nm} r^{-n} \rho_t r^{-k-1}\, dr \sim ct^{-n-k}, \quad \text{as } t \to 0,$$

$$(1, |\varphi_t|) = c\int_{\beta t}^{\gamma t} \rho_t r^{-k-1}\, dr = O(t^{-k}), \quad \text{as } t \to 0.$$

A fortiori, $d_{nm} = 0$ for $n \geqslant k$.

We have reduced u to

$$\begin{aligned} u = {} & A_{k1} S_{k1} + A_{k2} S_{k2} + d + \sum_{\substack{n=1 \\ n \neq k/2}}^{k-1} \sum_{m=1}^{2} A_{nm} r^{n-k} S_{nm} \\ & + r^{-k/2} \log r \sum_{m=1}^{2} a_{(k/2)m} S_{(k/2)m} + \sum_{n=1}^{k-1} \sum_{m=1}^{2} d_{nm} r^{-n} S_{nm} \\ & + a\tau(r) + bs(r) + c\sigma(r). \end{aligned}$$

Since u is bounded, we have, on letting $r \to \infty$, $A_{nm} = d_{nm} = 0$ for $n = 1, \ldots, k-1$, and $a_{(k/2)1} = a_{(k/2)2} = a = b = c = 0$. We infer that

$$u = A_{k1} S_{k1} + A_{k2} S_{k2} + d = A_{k1} \cos k\theta + A_{k2} \sin k\theta + d.$$

The proof of Theorem 1 is complete.

In the case $k = 2$, we have $\alpha = 0$, and therefore

COROLLARY [2]. *$H^2B(E_0^2)$ is generated by* 1, $\sin 2\theta$, *and* $\cos 2\theta$.

1.3. Generators of H^2B functions on E^3_α. We proceed to show

THEOREM 2. *If $\alpha = -1 \pm \sqrt{\frac{1}{2}k(k+1)}$, $k \in Z^+$, then $H^2B(E^3_\alpha)$ is generated by* 1 *and $\{S_{km}\}_{m=1}^{m_k}$, the set of fundamental spherical harmonics of order k.*

PROOF. Again we will give the proof only for $\alpha = -1 - \sqrt{\frac{1}{2}k(k+1)}$, the other case being similar. We start with

$$\begin{aligned} p_n &= \tfrac{1}{2}\Big\{-(3-2)(\alpha+1) + \big[(3-2)^2(\alpha+1)^2 + 4n(n+3-2)\big]^{1/2}\Big\}, \\ &= \tfrac{1}{4}\Big\{\sqrt{2k(k+1)} + \sqrt{2k(k+1) + 16n(n+1)}\Big\}, \\ q_n &= \tfrac{1}{4}\Big\{\sqrt{2k(k+1)} - \sqrt{2k(k+1) + 16n(n+1)}\Big\}. \end{aligned}$$

If $P_n = 0$, then $\frac{1}{2}N(\alpha+1) + p_n = 0$ or $n(n+1) = \frac{3}{8}k(k+1)$. For fixed k this is a quadratic in n which has at most one positive integral solution. If such a solution exists, we denote it by $n = \mu$. Observe that $\mu < k$.

If $Q_n = 0$, then

$$-\tfrac{3}{4}\sqrt{2k(k+1)} = -\tfrac{1}{4}\sqrt{2k(k+1)} + \tfrac{1}{4}\sqrt{2k(k+1) + 16n(n+1)},$$

which has no solution, so that there is no ν term in $u \in H^2B(E^3_\alpha)$.

Thus u has the representation

$$\begin{aligned} u = &\sum_{n\neq\mu}\sum_{m=1}^{m_n} A_{nm} r^{p_n+2\alpha+2} S_{nm} + \sum_{n=1}^{\infty}\sum_{m=1}^{m_n} B_{nm} r^{q_n+2\alpha+2} S_{nm} \\ &+ r^{(2-N/2)(\alpha+1)} \log r \cdot \sum_{m=1}^{m_\mu} a_{\mu m} S_{\mu m} + \sum_{n=1}^{\infty}\sum_{m=1}^{m_n} (c_{nm} r^{p_n} + d_{nm} r^{q_n}) S_{nm} \\ &+ a\tau(r) + bs(r) + c\sigma(r) + d. \end{aligned}$$

The remainder of the proof will be divided into Steps I–V.

I. For any given n, the highest order term is obviously $c_{nm} r^{p_n}$; we eliminate this first.

Choose $\rho \in C_0^\infty$, $\rho \geqslant 0$, $\operatorname{supp}\rho \subset (0,1)$. Define $\rho_t(r) = \rho(r-t)$, and $\varphi_t = \rho_t S_{nm}$. Then

$$\begin{aligned} (u, \varphi_t) = c\int_t^{t+1} \Big[&A_{nm} r^{p_n+2\alpha+2} + B_{nm} r^{q_n+2\alpha+2} \\ &+ a_{\mu m} r^{-1/4(2k(k+1))^{1/2}} \log r + c_{nm} r^{p_n} + d_{nm} r^{q_n}\Big] \\ &\cdot \rho_t r^{-1-3/2(2k(k+1))^{1/2}}\, dr, \end{aligned}$$

$$(1, |\varphi_t|) = c\int_t^{t+1} \rho_t r^{-1-3/2(2k(k+1))^{1/2}}\, dr = O(t^{-1-3/2(2k(k+1))^{1/2}}),$$

whereas

$$(u, \varphi_t) \sim c t^{p_n - 1 - 3/2(2k(k+1))^{1/2}}, \quad \text{as } t \to \infty.$$

Since $p_n > 0$ for all n, it follows that $c_{nm} = 0$ for all n and m.

II. For $n > k$, $p_n + 2\alpha + 2 > 0$, $p_n + 2\alpha + 2 > q_n$, and $p_n + 2\alpha + 2 > \mu$. After Step I above, the latter statements assure us that, for a fixed $n > k$, $A_{nm} r^{p_n + 2\alpha + 2}$ is the dominant term as $r \to \infty$. If $A_{nm} \neq 0$ for $n > k$, the test function from Step I gives

$$(u, \varphi_t) = c \int_t^{t+1} \left[A_{nm} r^{p_n + 2\alpha + 2} + \cdots \right] \rho_t r^{-1 - 3/2(2k(k+1))^{1/2}} \, dr$$

$$\sim c t^{p_n + 2\alpha + 2 - 1 - 3/2(2k(k+1))^{1/2}}, \quad \text{as } t \to \infty.$$

Thus $A_{nm} = 0$ for $n > k$.

III. Suppose $B_{nm} \neq 0$ for some n. Choose $\rho \in C_0^\infty$, $\rho \geqslant 0$, $\operatorname{supp} \rho \subset (\beta, \gamma)$, $0 < \beta < \gamma < 1$. Define $\rho_t(r) = \rho(r/t)$, $\varphi_t = \rho_t S_{nm}$. Then

$$(u, \varphi_t) = c \int_{\beta t}^{\gamma t} \left[B_{nm} r^{q_n + 2\alpha + 2} + \cdots \right] \rho_t r^{-1 - 3/2(2k(k+1))^{1/2}} \, dr$$

$$\sim c t^{q_n + 2\alpha + 2 - 3/2(2k(k+1))^{1/2}}, \quad \text{as } t \to 0,$$

$$(1, |\varphi_t|) = c \int_{\beta t}^{\gamma t} \rho_t r^{-1 - 3/2(2k(k+1))^{1/2}} \, dr = O\left(t^{-3/2(2k(k+1))^{1/2}}\right), \quad \text{as } t \to 0.$$

The relation $q_n + 2\alpha + 2 < 0$ for all n implies $B_{nm} = 0$ for all n and m.

IV. As in Step II, assume $n > k$. In view of the above, the test function from Step III gives

$$(u, \varphi_t) = c \int_{\beta t}^{\gamma t} d_{nm} r^{q_n} \rho_t r^{-1 - 3/2(2k(k+1))^{1/2}} \, dr \sim c t^{q_n - 3/2(2k(k+1))^{1/2}}$$

and

$$(1, |\varphi_t|) = c \int_{\beta t}^{\gamma t} \rho_t r^{-1 - 3/2(2k(k+1))^{1/2}} \, dr = O\left(t^{-3/2(2k(k+1))^{1/2}}\right), \quad \text{as } t \to 0.$$

Inasmuch as $q_n < 0$ for all n, $d_n = 0$ for all $n > k$.

V. We have now reduced u to

$$u = \sum_{\substack{n=1 \\ n \neq \mu}}^{k} \sum_{m=1}^{m_n} A_{nm} r^{p_n + 2\alpha + 2} S_{nm} + r^{-1/4(2k(k+1))^{1/2}} \log r \cdot \sum_{m=1}^{m_\mu} a_{\mu m} S_{\mu m}$$

$$+ \sum_{n=1}^{k} \sum_{m=1}^{m_n} d_{nm} r^{q_n} S_{nm} + a\tau(r) + bs(r) + c\sigma(r) + d.$$

As in the proof of Theorem 1, we may eliminate a finite number of linearly independent unbounded terms and are left with

$$u = \sum_{m=1}^{m_k} A_{km} S_{km} + d.$$

The proof of Theorem 2 is complete.

In particular, when $k = 1$, we have $\alpha = 0$, and the S_{1m} are $\cos\theta \cos\psi$, $\sin\theta \sin\psi$ and $\cos\theta$, where θ is the angle between the vector and the z-axis.

COROLLARY [2]. *$HB^2(E_0^3)$ is generated by* 1, $\cos\theta \cos\psi$, $\sin\theta \sin\psi$, *and* $\cos\theta$.

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