INTERPOLATION OF UNIFORMLY CONVEX BANACH SPACES

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Abstract. If $A_0$ and $A_1$ are a compatible couple of Banach spaces, one of which is uniformly convex, then the complex interpolation spaces $[A_0, A_1]_\theta$ are also uniformly convex for $0 < \theta < 1$. Estimates are given for the moduli of convexity and smoothness of $[A_0, A_1]_\theta$ in terms of these moduli for $A_0$ and $A_1$. In general, up to equivalence of moduli these estimates are best possible.

A result of Beauzamy [1, p. 71] states that if $A_0$ and $A_1$ are compatible Banach spaces, one of which is uniformly convex, then the spaces $S(p, \xi_0, A_0, \xi_1, A_1)$ obtained from $A_0$ and $A_1$ by the method of real interpolation, are also uniformly convex for $1 < p < \infty$. In this note we present an analogue of this result for the complex interpolation method and give an estimate for the modulus of convexity of $[A_0, A_1]_\theta$ in terms of the moduli of $A_0$ and $A_1$. This estimate is best possible in general, up to equivalence of moduli. We remark that for the power type of the moduli, such an estimate is implicit in [10, section 1.9]. We use the notation and definitions of [7] and of [3,2] for concepts from the geometry of Banach spaces and from interpolation space theory respectively. The letters $A, A_0, A_1$ will always denote Banach spaces over the complex field. $(A_0, A_1)$ is a compatible couple (interpolation pair). The complex interpolation space $[A_0, A_1]_\theta$ is denoted by $A_\theta$.

We say that a positive function $f$ on $\mathbb{R}^+$ is dominated by another function $g$ (notation: $f < g$ or $f(t) < g(t)$) if there are positive constants $a, b$ such that

$$f(t) \leq ag(bt) \quad \text{for all } t \in \mathbb{R}^+.$$ 

$f$ is equivalent to $g$ ($f \sim g$) if $f < g$ and $g < f$. We denote by $f^{-1}$ the inverse function of $f$ (if it exists). We denote the moduli of uniform convexity and uniform smoothness of $A$ by $\delta_A(\varepsilon)$ and $\rho_A(\tau)$ respectively.

It is known [4; 7, part II] that $\rho_A$ is an Orlicz function and that $\delta_A$ is equivalent to an Orlicz function $\delta_A$ ($\delta_A(\varepsilon)$ is defined only for $0 < \varepsilon < 2$ but $\delta_A$ is defined for all $\varepsilon > 0$ and equivalent to $\delta_A$ near 0). Also, $\rho_A(\tau) > 0$ for $\tau > 0$ (this can be concluded from the duality formula $\rho_A(\tau) = \frac{1}{2} \sup_{r>0}[r^2 - 2\delta_A(\varepsilon)]$ and the fact that $\delta_A(\varepsilon) < \varepsilon^2$) and being an Orlicz function, it turns out that $\rho_A$ is strictly increasing. $\delta_A$ is also strictly increasing if $A$ is uniformly convex ($\delta_A$ itself is strictly increasing when $A$ is uniformly convex since $\delta_A(\varepsilon)/\varepsilon$ is nondecreasing). If $L$ is a Banach lattice of measurable functions on a measure space $(\Omega, \Sigma, \mu)$ we denote by $L(A)$ the space of

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A-valued Bochner measurable functions $f$ such that $\|f\|_{\mathcal{A}} \in L$, normed by $\|f\|_{L(\mathcal{A})} = \|f\|_{\mathcal{A}} L$. For $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, we denote by $L_p^n$ or $L_p^n$ the space $\mathbb{R}^n$ with the norms $\|(a_j)\| = (\sum_{i=1}^{n} |a_i|^p)^{1/p}$ or, respectively,

$$\|(a_j)\| = \left( \frac{1}{n} \sum_{i=1}^{n} |a_i|^p \right)^{1/p}.$$

We conclude this introduction by recalling a construction of Calderón [3, 13.5]: If $M_0$ and $M_1$ are Banach lattices of functions on $(\Omega, \Sigma, \mu)$ and $0 < \theta < 1$ then the Banach lattice $M_0^{1-\theta} M_1^{\theta}$ is defined to be the space of all measurable functions $f$ which satisfy $|f| \leq \lambda |f_0|^{1-\theta} |f_1|^\theta$ for some $f_0 \in M_0, f_1 \in M_1, \lambda > 0$ with $\|f_i\|_{M_i} \leq 1, i = 0, 1$. The norm in $M_0^{1-\theta} M_1^{\theta}$ is the infimum of the values of the above $\lambda$.

Finally, we wish to thank the referee of a previous version of this paper for helpful remarks and in particular, for calling our attention to Lemma 3 which enabled us to provide a simpler proof and obtain a stronger result.

**Theorem 1.** Let $(A_0, A_1)$ be a compatible couple of Banach spaces and $0 < \theta < 1$.

(i) If at least one of $A_0, A_1$ (e.g. $A_0$) is uniformly convex, then $A_\theta = [A_0, A_1]_\theta$ is uniformly convex and

$$\delta_{A_\theta}(\epsilon) > \delta_{A_0}(\epsilon^{1/(1-\theta)})$$

(also, $\delta_{A_\theta}(\epsilon) > \delta_{A_1}(\epsilon^{1/\theta})$ if $A_1$ is uniformly convex).

(ii) If both $A_0$ and $A_1$ are uniformly convex, then $\delta_{A_\theta} > \delta$ where $\delta$ is the inverse function of $(\delta_{A_0}^{-1})^{1-\theta}(\delta_{A_1}^{-1})^\theta$.

(iii) $\rho_A < \rho$, where $\rho$ is the inverse function of $(\rho_{A_0}^{-1})^{1-\theta}(\rho_{A_1}^{-1})^\theta$.

We remark that simple examples (e.g. $L_p$-spaces) show that the above estimates are, in general, best possible up to equivalence of moduli.

By duality between $\delta_{A_\theta}$ and $\rho_{A_\theta}, \delta_{A_\theta}$ and $\rho_{A_\theta}$ (up to equivalence of functions), (iii) follows from (i) and (ii). For the proof of (i) and (ii) we introduce the following definitions and lemmas.

For a strictly increasing Orlicz function $\phi$, we define the lattice norm $\| \|_{\phi}$ on $\mathbb{R}^2$ by

$$\|(a, b)\|_{\phi} = \inf \{ \rho > 0 \mid |a|/\rho + \phi(|b|/\rho) \leq 1 \},$$

which is the same as

$$\|(a, b)\|_{\phi} = \inf \{ \lambda > 0 \mid \exists (c, d), |c| + |d| \leq 1, |a| \leq \lambda |c|, |b| \leq \lambda \phi^{-1}(|d|) \}.$$

We denote the Banach lattice $(\mathbb{R}^2, \| \|_{\phi})$ by $M_\phi$.

**Lemma 2.** Let $\phi_0, \phi_1$ be two strictly increasing Orlicz functions and $\phi = [(\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^\theta]^{-1}$. Then

$$\| \|_{\phi(a \cdot)} \leq \| \|_{M_0^{1-\theta} M_1^{\theta}} \leq \| \|_{\phi}$$

where $\alpha = \min(\theta, 1 - \theta)$.
Proof. It is clear that $\phi$ is an Orlicz function.

Since $\phi_i^{-1}$ are concave and $\phi_i^{-1}(0) = 0$ ($i = 0, 1$), we have for $0 < \lambda \leq 1$ and $t \geq 0$

$$\lambda \phi_i^{-1}(t) \leq \phi_i^{-1}(\lambda t).$$

Consequently, if $0 < t_1 \leq t_2$ we have

$$\phi_i^{-1}((1 - \theta)t_1 + \theta t_2) \geq \frac{(1 - \theta)t_1 + \theta t_2}{t_2} \phi_i^{-1}(t_2) \geq \theta \phi_i^{-1}(t_2) \geq \theta \phi_i^{-1}(t_1).$$

In a similar way, for $0 < t_2 \leq t_1$ we have

$$\phi_i^{-1}((1 - \theta)t_1 + \theta t_2) \geq (1 - \theta) \phi_i^{-1}(t_1) \geq (1 - \theta) \phi_i^{-1}(t_2),$$

hence for all $t_1, t_2 \geq 0$

$$\bigl[\phi_0^{-1}(t_1)\bigr]^{1-\theta} \bigl[\phi_1^{-1}(t_2)\bigr]^{\theta} \leq \alpha^{-1} \bigl[\phi_0^{-1}((1 - \theta)t_1 + \theta t_2)\bigr]^{1-\theta} \bigl[\phi_1^{-1}((1 - \theta)t_1 + \theta t_2)\bigr]^{\theta}$$

with $\alpha = \min(\theta, 1 - \theta)$.

Now suppose $\|(a, b)\|_{M_t^{-t}M_t^2} \leq 1$. By the definition, there are $c_1, c_2, d_1, d_2$ with $|c_j| + |d_j| \leq 1, j = 1, 2$, such that

$$|a| \leq |c_1|^{1-\theta} |c_2|^\theta$$

and

$$|b| \leq \bigl[\phi_0^{-1}(|d_1|)\bigr]^{1-\theta} \bigl[\phi_1^{-1}(|d_2|)\bigr]^{\theta}$$

then

$$|a| \leq (1 - \theta) |c_1| + \theta |c_2|$$

and

$$|b| \leq \alpha^{-1} \bigl[\phi_0^{-1}((1 - \theta) |d_1| + \theta |d_2|)\bigr]^{1-\theta} \bigl[\phi_1^{-1}((1 - \theta) |d_1| + \theta |d_2|)\bigr]^{\theta}$$

or

$$\alpha |b| \leq \phi^{-1}((1 - \theta) |d_1| + \theta |d_2|).$$

Since

$$|(1 - \theta) |c_1| + \theta |c_2| | + |(1 - \theta) |d_1| + \theta |d_2| | \leq 1$$

this implies that $\|(a, b)\|_{\phi(|a|)} \leq 1$ thus proving the left-hand side of (1). The right-hand side of (1) is straightforward.

Lemma 3. $\delta_A > \delta$ if and only if there are positive constants $a, b$ such that for all $x, y$ in $A$ holds

$$(2) \quad \frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1 \Rightarrow \|x\| + a \delta(b \|y\|) \leq 1.$$

Proof. If (2) holds, then obviously $\delta_A > \delta$ (take $z, w$ with $\|z\|, \|w\| \leq 1$ and write $z = x + y, w = x - y$).

On the other hand, by [4], ([5]) $\delta_A \sim \delta_{L_2(A)}$. Assume $\delta_{L_2(A)} > \delta$. Given that

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1$$
we consider $x$ as a constant function in $L_2(A)$ over a probability space and look at the monotone basic sequence of two elements $\{x, \varepsilon y\}$ in $L_2(A)$, where $\varepsilon$ is a Bernoulli variable. This sequence satisfies the requirement of Proposition 2.1 in [8] hence, by that proposition

$$\|x\| + \delta_{L_2(A)}(\|y\|) \leq 1.$$  

**Proof of Theorem 1.** We assume, as we may, that $\delta_i$ are Orlicz functions, strictly increasing if $A_i$ is uniformly convex.

We prove first (ii). By Lemma 3, (2) is satisfied by $A_i, \delta_i$, with constants $a_i, b_i$ ($i = 0, 1$). If $\phi_i = a_i \delta_i(b_i)$ then this means that the operator

$$T: A_i \times A_i \to A_i \times A_i$$

defined by $T(z, w) = ((z + w)/2, (z - w)/2)$, has norm $\leq 1$ as an operator

$$T: L^2(A_i) \to M_{\phi_i}(A_i)$$

for both $i$'s.

Hence $T$ has norm $\leq 1$ as an operator

$$T: L^2(A_\theta) \to [M_{\phi_0}(A_0), M_{\phi_1}(A_1)]_\theta.$$

The lattice $M_{\phi_0}^{-1-\theta} M_{\phi_1}^\theta$ clearly has the convergence property required for the result in paragraph (i) of [3, 13.6] and that result yields that

$$[M_{\phi_0}(A_0), M_{\phi_1}(A_1)]_\theta = M_{\phi_0}^{-1-\theta} M_{\phi_1}^\theta(A_\theta)$$

which, together with Lemma 2 yields that $T$ has norm $\leq 1$ as an operator

$$T: L^2(A_\theta) \to M_{\phi(\cdot)}(A_\theta)$$

where $\phi = [(\phi_0^{-1})^{1-\theta}(\phi_1)^{1-\theta}]^{-1}$.

The last assertion is just (2) for $A = A_\theta, \delta = \phi, a = 1, b = \alpha$ from which (ii) follows easily.

The proof of (i) goes along the same lines, we use (2) for $A_0, \delta_{A_0}, a_0, b_0$ and take into account that in every Banach space we have

$$\left(\frac{\|x + y\|^2 + \|x - y\|^2}{2}\right)^{1/2} \geq \max(\|x\|, \|y\|)$$

i.e. $T: L^2(A_1) \to L^2(A_1)$ has norm $\leq 1$.

Hence $T: L^2(A_\theta) \to [M_{\phi_0}(A_0), L^2(A_1)]_\theta$ has norm $\leq 1$; this yields for $x, y \in A_\theta$

$$\frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1 \Rightarrow \|x\|^{1/(1-\theta)} + \phi_0(\|y\|^{1/(1-\theta)}) \leq 1;$$

hence, if $z, w \in A_\theta, \|z\|, \|w\| < 1$ then

$$\frac{z + w}{2} \leq \left[1 - \phi_0\left(\frac{\|z - w\|^{1/(1-\theta)}}{2}\right)\right]^{1-\theta}$$

$$\leq 1 - (1 - \theta)\phi_0\left(\frac{\|z - w\|^{1/(1-\theta)}}{2}\right)$$

which proves (i). $\Box$
Remark. Another approach which can be used to prove similar results proceeds via use of the martingale characterization of uniform convexity of [8 and 6]. A disadvantage of this approach is that in the general case it yields only isomorphic results. However, for functions $\delta(e)$ of the form $e^q$ it gives the same result as here, and in this case it works also for the real interpolation method. Namely, if $A_i$ are isomorphic to spaces with moduli of uniform convexity $> e^q$, then $(A_0, A_1)_{\theta,s}$ is isomorphic to a space with modulus of convexity $> e^s$ for every $s$ with $q_0 \leq s < \infty$ $(1/q_0 = (1 - \theta)/q_0 + \theta/q_1)$.

References


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