INTERPOLATION OF UNIFORMLY CONVEX BANACH SPACES

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Abstract. If $A_0$ and $A_1$ are a compatible couple of Banach spaces, one of which is uniformly convex, then the complex interpolation spaces $[A_0, A_1]_\theta$ are also uniformly convex for $0 < \theta < 1$. Estimates are given for the moduli of convexity and smoothness of $[A_0, A_1]_\theta$ in terms of these moduli for $A_0$ and $A_1$. In general, up to equivalence of moduli these estimates are best possible.

A result of Beauzamy [1, p. 71] states that if $A_0$ and $A_1$ are compatible Banach spaces, one of which is uniformly convex, then the spaces $S(p, \xi_0, A_0, \xi_1, A_1)$ obtained from $A_0$ and $A_1$ by the method of real interpolation, are also uniformly convex for $1 < p < \infty$. In this note we present an analogue of this result for the complex interpolation method and give an estimate for the modulus of convexity of $[A_0, A_1]_\theta$ in terms of the moduli of $A_0$ and $A_1$. This estimate is best possible in general, up to equivalence of moduli. We remark that for the power type of the moduli, such an estimate is implicit in [10, section 1.9]. We use the notation and definitions of [7] and of [3,2] for concepts from the geometry of Banach spaces and from interpolation space theory respectively. The letters $A, A_0, A_1$ will always denote Banach spaces over the complex field. $(A_0, A_1)$ is a compatible couple (interpolation pair). The complex interpolation space $[A_0, A_1]_\theta$ is denoted by $A_\theta$.

We say that a positive function $f$ on $\mathbb{R}^+$ is dominated by another function $g$ (notation: $f < g$ or $f(t) < g(t)$) if there are positive constants $a, b$ such that

$$f(t) \leq ag(bt) \quad \text{for all } t \in \mathbb{R}^+.$$ 

$f$ is equivalent to $g$ ($f \sim g$) if $f < g$ and $g < f$. We denote by $f^{-1}$ the inverse function of $f$ (if it exists). We denote the moduli of uniform convexity and uniform smoothness of $A$ by $\delta_A(\varepsilon)$ and $\rho_A(\tau)$ respectively.

It is known [4; 7, part II] that $\rho_A$ is an Orlicz function and that $\delta_A$ is equivalent to an Orlicz function $\delta_A(\varepsilon)$ is defined only for $0 < \varepsilon < 2$ but $\delta_A$ is defined for all $\varepsilon > 0$ and equivalent to $\delta_A$ near 0). Also, $\rho_A(\tau) > 0$ for $\tau > 0$ (this can be concluded from the duality formula $\rho_A(\tau) = \frac{1}{2} \sup_{\varepsilon > 0} \{\varepsilon - 2\delta_A(\varepsilon)\}$ and the fact that $\delta_A(\varepsilon) < \varepsilon^2$) and being an Orlicz function, it turns out that $\rho_A$ is strictly increasing. $\delta_A$ is also strictly increasing if $A$ is uniformly convex ($\delta_A$ itself is strictly increasing when $A$ is uniformly convex since $\delta_A(\varepsilon)/\varepsilon$ is nondecreasing). If $L$ is a Banach lattice of measurable functions on a measure space $(\Omega, \Sigma, \mu)$ we denote by $L(A)$ the space of
$A$-valued Bochner measurable functions $f$ such that $\|f\|_A \in L$, normed by $\|f\|_{L(A)} = \|f\|_A \|f\|_L$. For $1 \leq p < \infty$ and $n \in \mathbb{N}$, we denote by $L_p^n$ or $L_p^n(\mu)$ the space $\mathbb{R}^n$ with the norms $\|(a_i)\| = (\sum_{i=1}^n |a_i|^p)^{1/p}$ or, respectively,

$$\|(a_i)\| = \left(\frac{1}{n} \sum_{i=1}^n |a_i|^p\right)^{1/p}.$$

We conclude this introduction by recalling a construction of Calderón [3, 13.5]: If $M_0$ and $M_1$ are Banach lattices of functions on $(\Omega, \Sigma, \mu)$ and $0 < \theta < 1$ then the Banach lattice $M_0^{1-\theta}M_1^\theta$ is defined to be the space of all measurable functions $f$ which satisfy $|f| \leq \lambda |f_0|^{1-\theta} |f_1|^\theta$ for some $f_0 \in M_0, f_1 \in M_1, \lambda > 0$ with $\|f_i\| \leq 1, i = 0, 1$. The norm in $M_0^{1-\theta}M_1^\theta$ is the infimum of the values of the above $\lambda$.

Finally, we wish to thank the referee of a previous version of this paper for helpful remarks and in particular, for calling our attention to Lemma 3 which enabled us to provide a simpler proof and obtain a stronger result.

**Theorem 1.** Let $(A_0, A_1)$ be a compatible couple of Banach spaces and $0 < \theta < 1$.

(i) If at least one of $A_0, A_1$ (e.g. $A_0$) is uniformly convex, then $A_\theta = [A_0, A_1]_\theta$ is uniformly convex and

$$\delta_{A_\theta}(\varepsilon) > \delta_{A_0}(e^{\varepsilon/(1-\theta)})$$

(also, $\delta_{A_\theta}(\varepsilon) > \delta_{A_1}(e^{\varepsilon/\theta})$ if $A_1$ is uniformly convex).

(ii) If both $A_0$ and $A_1$ are uniformly convex, then $\delta_{A_\theta} > \delta$ where $\delta$ is the inverse function of $(\delta_{A_0}^{-1})^{-\theta}(\delta_{A_1}^{-1})^\theta$.

(iii) $\rho_{A_\theta} < \rho$, where $\rho$ is the inverse function of $(\rho_{A_0}^{-1})^{-\theta}(\rho_{A_1}^{-1})^\theta$.

We remark that simple examples (e.g. $L_p$-spaces) show that the above estimates are, in general, best possible up to equivalence of moduli.

By duality between $\delta_\theta$ and $\rho_\phi$, $\delta_{A_\phi}$ and $\rho_\phi$ (up to equivalence of functions), (iii) follows from (i) and (ii). For the proof of (i) and (ii) we introduce the following definitions and lemmas.

For a strictly increasing Orlicz function $\phi$, we define the lattice norm $\|(a, b)\|$ on $\mathbb{R}^2$ by

$$\|(a, b)\|_\phi = \inf\{\rho > 0 \mid a/\rho + \phi(|b|/\rho) \leq 1\},$$

which is the same as

$$\|(a, b)\|_\phi = \inf\{\lambda > 0 \exists (c, d), |c| + |d| \leq 1, |a| \leq \lambda |c|, |b| \leq \lambda \phi^{-1}(|d|)\}.$$ 

We denote the Banach lattice $(\mathbb{R}^2, \|(a, b)\|_\phi)$ by $M_\phi$.

**Lemma 2.** Let $\phi_0, \phi_1$ be two strictly increasing Orlicz functions and $\phi = [(\phi_0^{-1})^{-\theta}(\phi_1^{-1})^\theta]^{-1}$. Then

$$\|(a, b)\|_\phi \leq \|(a, b)\|_{M_0^{-\theta}M_1^\theta} \leq \|(a, b)\|_\phi$$

where $\alpha = \min(\theta, 1 - \theta)$. 

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Proof. It is clear that $\phi$ is an Orlicz function.
Since $\phi_i^{-1}(0) = 0$ $(i = 0, 1)$, we have for $0 < \lambda < 1$ and $t > 0$
$$
\lambda \phi_i^{-1}(t) \leq \phi_i^{-1}(\lambda t).
$$
Consequently, if $0 < t_1 \leq t_2$ we have
$$
\phi_i^{-1}((1 - \theta)t_1 + \theta t_2) \geq \frac{(1 - \theta)t_1 + \theta t_2}{t_2} \phi_i^{-1}(t_2) \geq \theta \phi_i^{-1}(t_1) \geq \theta \phi_i^{-1}(t_2).
$$
In a similar way, for $0 < t_2 \leq t_1$ we have
$$
\phi_i^{-1}((1 - \theta)t_1 + \theta t_2) \geq (1 - \theta) \phi_i^{-1}(t_1) \geq (1 - \theta) \phi_i^{-1}(t_2),
$$
hence for all $t_1, t_2 \geq 0$
$$
[\phi_0^{-1}(t_1)]^{1 - \theta} \phi_1^{-1}(t_2) \leq \alpha^{-1}[\phi_0^{-1}((1 - \theta)t_1 + \theta t_2)]^{1 - \theta} \phi_1^{-1}((1 - \theta)t_1 + \theta t_2)
$$
with $\alpha = \min(\theta, 1 - \theta)$.

Now suppose $\|(a, b)\|_{\mathcal{M}_1^{-1}-\mathcal{M}_2} \leq 1$. By the definition, there are $c_1, c_2, d_1, d_2$ with
$$
|c_j| + |d_j| \leq 1, j = 1, 2, \text{ such that }
$$
$$
|a| \leq |c_1|^{-\theta} |c_2|^\theta
$$
and
$$
|b| \leq [\phi_0^{-1}(|d_1|)]^{1 - \theta} [\phi_1^{-1}(|d_2|)]^\theta
$$
then
$$
|a| \leq (1 - \theta) |c_1| + \theta |c_2|
$$
and
$$
|b| \leq \alpha^{-1}[\phi_0^{-1}((1 - \theta) |d_1| + \theta |d_2|)]^{1 - \theta} [\phi_1^{-1}((1 - \theta) |d_1| + \theta |d_2|)]^\theta
$$
or
$$
\alpha |b| \leq \phi_1^{-1}((1 - \theta) |d_1| + \theta |d_2|).
$$
Since
$$
[(1 - \theta) |c_1| + \theta |c_2|] + [(1 - \theta) |d_1| + \theta |d_2|] \leq 1
$$
this implies that $\|(a, b)\|_{\mathcal{M}_1} \leq 1$ thus proving the left-hand side of (1). The right-hand side of (1) is straightforward.

Lemma 3. $\delta_A > \delta$ if and only if there are positive constants $a, b$ such that for all $x, y$ in $A$ holds

\begin{equation}
\|x + y\|^2 + \|x - y\|^2 \leq \frac{2}{2} \Rightarrow \|x\| + a \delta(b \|y\|) \leq 1.
\end{equation}

Proof. If (2) holds, then obviously $\delta_A > \delta$ (take $z, w$ with $\|z\|, \|w\| \leq 1$ and write $z = x + y, w = x - y$).

On the other hand, by [4], [5]) $\delta_A \sim \delta_{L_2(A)}$. Assume $\delta_{L_2(A)} > \delta$. Given that
\begin{equation}
\|x + y\|^2 + \|x - y\|^2 \leq \frac{2}{2} \Rightarrow \|x\| + \delta(b \|y\|) \leq 1
\end{equation}

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we consider \( x \) as a constant function in \( L_2(A) \) over a probability space and look at the monotone basic sequence of two elements \( \{x, \varepsilon y\} \) in \( L_2(A) \), where \( \varepsilon \) is a Bernoulli variable. This sequence satisfies the requirement of Proposition 2.1 in [8] hence, by that proposition
\[
\|x\| + \delta_{L_2(A)}(\|y\|) \leq 1.
\]

**Proof of Theorem 1.** We assume, as we may, that \( \delta_{\epsilon_i} \) are Orlicz functions, strictly increasing if \( A_i \) is uniformly convex.

We prove first (ii). By Lemma 3, (2) is satisfied by \( A_i, \delta_{\epsilon_i} \) with constants \( a_i, b_i \) \((i = 0, 1)\). If \( \phi_i = a_i \delta_{\epsilon_i}(b_i) \) then this means that the operator
\[
T: A_i \times A_i \to A_i \times A_i
\]
defined by \( T(z, w) = ((z + w)/2, (z - w)/2) \), has norm \( \leq 1 \) as an operator
\[
T: L_2^i(A_i) \to M_{\phi_i}(A_i)
\]
for both \( i \)'s.

Hence \( T \) has norm \( \leq 1 \) as an operator
\[
T: L_2^i(A_0) \to M_{\phi_i}(A_0) \equiv M_{\phi_i}(A_i).
\]

The lattice \( M_{\phi_0}^{-\theta}M_{\phi_1}^\theta \) clearly has the convergence property required for the result in paragraph (i) of [3, 13.6] and that result yields that
\[
\left[ M_{\phi_0}(A_0), M_{\phi_i}(A_1) \right]_\theta = M_{\phi_0}^{-\theta}M_{\phi_1}^{\theta}(A_\theta)
\]
which, together with Lemma 2 yields that \( T \) has norm \( \leq 1 \) as an operator
\[
T: L_2^i(A_\theta) \to M_{\phi(a_\theta)}(A_\theta)
\]
where \( \phi = [(\phi_0^{-1})^{1-\theta}(\phi_1^{-1})^{\theta}]^{-1} \).

The last assertion is just (2) for \( A = A_\theta, \delta = \phi, a = 1, b = a \) from which (ii) follows easily.

The proof of (i) goes along the same lines, we use (2) for \( A_0, \delta_{A_\theta}, a_0, b_0 \) and take into account that in every Banach space we have
\[
\left( \|x + y\|^2 + \|x - y\|^2 \right)^{1/2} \geq \max(\|x\|, \|y\|)
\]
i.e. \( T: L_2^i(A_i) \to l_\infty^i(A_1) \) has norm \( \leq 1 \).

Hence \( T: L_2^i(A_\theta) \to [M_{\phi_0}(A_0), l_\infty^i(A_1)]_\theta \) has norm \( \leq 1 \); this yields for \( x, y \in A_\theta \)
\[
\frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq 1 \Rightarrow \|x\|^{1/(1-\theta)} + \phi_0(\|y\|^{1/(1-\theta)}) \leq 1;
\]

hence, if \( z, w \in A_\theta, \|z\|, \|w\| \leq 1 \) then
\[
\frac{z + w}{2} \leq 1 - \phi_0(\frac{z - w}{2})^{1/(1-\theta)} \leq 1 - (1 - \theta)\phi_0(\frac{z - w}{2})^{1/(1-\theta)}
\]

which proves (i). \( \square \)
Remark. Another approach which can be used to prove similar results proceeds via use of the martingale characterization of uniform convexity of [8 and 6]. A disadvantage of this approach is that in the general case it yields only isomorphic results. However, for functions δ(e) of the form ε^θ it gives the same result as here, and in this case it works also for the real interpolation method. Namely, if A_j are isomorphic to spaces with moduli of uniform convexity > ε^{θ_j}, then (A_0, A_1)_θ, s is isomorphic to a space with modulus of convexity > ε^{θ} for every s with q_θ ≤ s < ∞ (1/q_θ = (1 − θ)/q_0 + θ/q_1).

References