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UNDECIDABLE THEORIES IN STATIONARY LOGIC

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ABSTRACT. It is shown that the theories of the following classes become undecidable in stationary logic: Well orderings with one unary predicate, linear orderings, and Boolean algebras. This is done by interpreting the theory of symmetric reflexive graphs.

1. Introduction. This paper establishes that the $L(aa)$-theories of well orderings with a distinguished subset (WoP), linear orderings (Lin) and Boolean algebras (Ba) are all undecidable. This result is of interest since some decidability results (e.g., Abelian groups, reals [4], and ordinals [7] extend from L to $L(Q)$ and then to $L(aa)$). It is known that the $L(Q)$-theories of Lin, Ba [2] and WoP (can be shown similar to the decidability of the $L(Q)$-theory of Lin) are decidable. Also the $L(aa)$-theory of ω_1 with a distinguished subset is decidable. This is a special case of the decidability of the monadic theory of ω_1 [3].

2. Definitions and notations. We shall implicitly assume throughout the paper that the structures we consider are uncountable. For a model $\mathfrak{M}$ for L its universe is denoted by $|\mathfrak{M}|$. Let $P_1(\mathfrak{M})$ denote the set of all countable subsets of $|\mathfrak{M}|$. A set $X \subseteq P_1(\mathfrak{M})$ is *unbounded*, if every $s_0 \in P_1(\mathfrak{M})$ is a subset of some $s \in X$. X is *closed*, if the union of each increasing sequence $s_0 \subseteq s_1 \subseteq \cdots \subseteq s_n \subseteq \cdots$ of elements of X is again an element of X. A set X is *cub*, if it is closed and unbounded. $\mathfrak{F}(\mathfrak{M})$ denotes the filter on $P_1(\mathfrak{M})$ generated by all cub sets. We define

$$\mathfrak{M} \models aa\, s\varphi(s) \quad \text{iff} \quad \{s \in P_1(\mathfrak{M}) : \mathfrak{M} \models \varphi(s)\} \in \mathfrak{F}(\mathfrak{M}).$$

If $\mathfrak{M} \models aa\, s\varphi(s)$, we say that property $\varphi(s)$ holds for *almost all* countable s.

We assume that the reader is familiar with [1]. In that paper it is shown that the quantifier Q_1, read "there exist uncountably many", is definable in $L(aa)$ by the formula

$$\text{stat}\, s \exists x(\varphi(x) \wedge \neg s(x)).$$

Thus $L(Q_1)$ is a sublogic of $L(aa)$.

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Let T be a theory, $\mathfrak{M} \models T$ and $\varphi(x_1,\ldots,x_n)$, $\psi(s)$ formulas of the corresponding $L(aa)$-language for T. Then we set

$$\varphi^{\mathfrak{M}} = \{\vec{a} \in |\mathfrak{M}|^n : \mathfrak{M} \models \varphi(\vec{a})\};$$
$$\psi^{\mathfrak{M}} = \{s \in P_1(\mathfrak{M}) : \mathfrak{M} \models \psi(s)\}.$$

If T is the elementary theory of some class of structures, then $T(aa)$ denotes the corresponding $L(aa)$-theory of the same class of models.

3. Preliminaries. Let Gr be the theory of infinite symmetric reflexive graphs. In order to show the undecidability of any theory T it is enough to show that Gr is interpretable in T, i.e. there are formulas $\varphi_0(x)$, $\varphi_1(x, y)$ and $\psi(x, y)$ such that

(*) **for each countable symmetric graph $\mathfrak{G} = \langle \omega, R \rangle$ there is a model $\mathfrak{M} \models T$ such that $\psi^{\mathfrak{M}}$ is a congruence relation on $\langle \varphi_0^{\mathfrak{M}}, \varphi_1^{\mathfrak{M}} \rangle$ and $\langle \varphi_0^{\mathfrak{M}}, \varphi_1^{\mathfrak{M}} \rangle / \psi^{\mathfrak{M}} \cong \mathfrak{G}$.**

Here we use the fact that the theory of infinite symmetric reflexive graphs is hereditarily undecidable. Let T^* be T together with a sentence ρ which says of a model $\mathfrak{M} \models T$, that $\langle \varphi_0^{\mathfrak{M}}, \varphi_1^{\mathfrak{M}} \rangle / \psi^{\mathfrak{M}}$ is a symmetric reflexive graph. So T^* and hence T is undecidable.

A proof of the following well-known theorem can be found in [**6**]:

THEOREM 1 (ULAM). *There is a set $\{S_\alpha : \alpha < \omega_1\}$ of pairwise disjoint stationary subsets of ω_1.*

4. Undecidability proofs. We set

$$\sigma(s, x, y) \equiv \forall z(s(z) \wedge z < x \rightarrow z \leqslant y)$$
$$\wedge \forall z(z < y \rightarrow \exists u(s(u) \wedge z < u \wedge u < y \wedge u < x)).$$

Thus, if $\mathfrak{M}$ is an ordering, $s \subseteq |\mathfrak{M}|$ and $a, b \in |\mathfrak{M}|$, and $\mathfrak{M} \models \sigma(s, a, b)$, then $b = \sup\{c \in |\mathfrak{M}| : s(c) \wedge c < a\}$.

LEMMA 2. *Let $A, B \subseteq \omega_1$, α, β, γ ordinals with $\alpha \neq \beta$, $\omega_1(\alpha + 1)$, $\omega_1(\beta + 1) < \gamma$, $P = \{\omega_1 \cdot \alpha + \delta : \delta \in A\} \cup \{\omega_1 \cdot \beta + \delta : \delta \in B\}$, $\mathfrak{M} = \langle \gamma, <, P \rangle$. Then*

$$\text{stat}\, s \exists xy(\sigma(s, \omega_1(\alpha + 1), x) \wedge \sigma(s, \omega_1(\beta + 1), y) \wedge P(x) \wedge P(y))$$

iff $A \cap B$ is stationary.

PROOF. For $\mathfrak{X} \in P_1(\mathfrak{M})$ let $\mathfrak{X}^* = \{s \in \mathfrak{X}$: for all $\delta < \omega_1$, $\omega_1 \cdot \alpha + \delta \in s$ iff $\omega_1 \cdot \beta + \delta \in s\}$. Let

$$\varphi(s) \equiv \exists xy(\sigma(s, \omega_1(\alpha + 1), x) \wedge \sigma(s, \omega_1(\beta + 1), y) \wedge P(x) \wedge P(y)).$$

Assume $\mathfrak{M} \models \text{stat}\, s\varphi(s)$. Let $C \subseteq \omega_1$. We set

$$\mathfrak{X}_C = \left\{ \begin{array}{l} s \in P_1(\mathfrak{M}) : \text{there is } \delta \in C \text{ with} \\ \sigma(s, \omega_1(\alpha + 1), \omega_1 \cdot \alpha + \delta) \text{ and} \\ \sigma(s, \omega_1(\beta + 1), \omega_1 \cdot \beta + \delta) \end{array} \right\}.$$

If C is closed unbounded, then $\mathfrak{X}_C \in \mathfrak{F}(\mathfrak{M})$. Assume $A \cap B$ is not stationary. Then there is a closed unbounded subset C of ω_1 with $C \cap (A \cap B) = \varnothing$. Then $\mathfrak{X}_C \in \mathfrak{F}(\mathfrak{M})$ but for no $s \in \mathfrak{X}_C$, $\mathfrak{M} \models \varphi(s)$. Thus $\mathfrak{M} \models \neg \text{stat}\, s\varphi(s)$. Assume on the other hand that $A \cap B$ is stationary. Let $\mathfrak{X} \in \mathfrak{F}(\mathfrak{M})$. We set

$$\mathfrak{X}^* = \{s \in \mathfrak{X}\text{: there is } \delta < \omega_1 \text{ with } \sigma(s, \omega_1(\alpha + 1), \omega_1 \cdot \alpha + \delta) \\ \text{and } \sigma(s, \omega_1(\beta + 1), \omega_1 \cdot \beta + \delta)\}.$$

Then $\mathfrak{X}^* \in \mathfrak{F}(\mathfrak{M})$. We set

$$X = \{\delta < \omega_1\text{: there is } s \in \mathfrak{X}^* \text{ with } \sigma(s, \omega_1(\alpha + 1), \omega_1 \cdot \alpha + \delta)\}.$$

Then X is a closed unbounded subset of ω_1 and thus there is $\delta \in X \cap A \cap B$. But then there is $s \in \mathfrak{X}^*$ with $\mathfrak{M} \models \varphi(s)$ and thus $\mathfrak{M} \models \text{stat}\, s\varphi(s)$. □

THEOREM 3. WoP(aa) *is undecidable.*

PROOF. We set

$$\varphi_0(x) \equiv \exists y(y < x) \wedge \forall z(z < x \to Q_1 u(z < u < x));$$
$$\varphi_1(x, y) \equiv \varphi_0(x) \wedge \varphi_0(y) \wedge \text{stat}\, s\exists zu(\sigma(s, x, z) \wedge \sigma(s, y, u) \wedge P(z) \wedge P(u));$$
$$\psi(x, y) \equiv x = y.$$

Let $\mathfrak{G} = \langle \omega, R \rangle$ be a countable symmetric reflexive graph. Let $\{S_n\colon n < \omega\}$ be a set of pairwise disjoint subsets of ω_1 and let $h\colon \omega \times \omega \to \omega$ be a 1-1 function. For $k < \omega$ let

$$A_k = \bigcup \{S_{h(k,i)}\colon (k, i) \in R\} \cup \bigcup \{S_{h(i,k)}\colon (k,i) \in R\}.$$

Then each A_k is a stationary set and $A_k \cap A_i$ is stationary iff $(k, i) \in R$. We set

$$P = \bigcup_{k<\omega} \{\omega_1 \cdot k + \alpha\colon \alpha \in A_k\}, \quad \mathfrak{M} = \langle \omega_1 \cdot \omega, <, P \rangle.$$

Then Lemma 2 implies immediately that

$$\langle \varphi_0^{\mathfrak{M}}, \varphi_1^{\mathfrak{M}} \rangle \cong \mathfrak{G}.$$

$\mathfrak{M}$ is an $\aleph_1$*-like dense linear ordering* if card$(|\mathfrak{M}|) = \aleph_1$ and for each $a \in |\mathfrak{M}|$, the set $\{b \in |\mathfrak{M}|\colon b < a\}$ forms an order which is isomorphic to η or $1 + \eta$. For each $A \subseteq \omega_1$ let $\Phi(A)$ be the $\aleph_1$-like dense linear ordering $1 + \eta + \sum_{\alpha<\omega_1} \tau_\alpha$ where

$$\tau_\alpha = \begin{cases} 1 + \eta & \text{for } \alpha \in A; \\ \eta & \text{otherwise.} \end{cases}$$

A proof of the following lemma can be found in [5]:

LEMMA 4 (CONWAY). *For each $\aleph_1$-like dense linear ordering $\mathfrak{M}$ with first element there exists $A \subseteq \omega_1$ with $\Phi(A) \cong \mathfrak{M}$. For $A, B \subseteq \omega_1$, $\Phi(A) \cong \Phi(B)$ iff there is some closed unbounded set $C \subseteq \omega_1$ with $A \cap C = B \cap C$.*

We use this lemma to show

THEOREM 5. Lin(aa) *is undecidable.*

PROOF. Let $\mathfrak{G}$ and A_k $(k < \omega)$ be as in the proof of Theorem 3. Let $\mathfrak{M} = \sum_{i<\omega} \Phi(A_i)$ and let a_i be the least element of $\Phi(A_{i+1})$. Let

$$\varphi_0(x) \equiv \exists y(y < x) \wedge \forall y(y < x \to Q_1 z(y < z \wedge z < x)).$$

Then $\varphi_0^{\mathfrak{M}} = \{a_i \colon i < \omega\}$. Let

$$\varphi_1(x, y) \equiv \text{stat } s \exists wz(\sigma(s, x, w) \wedge \sigma(s, y, z)).$$

As in Lemma 2 we have $\mathfrak{M} \models \varphi_1[a_i, a_k]$ iff $A_i \cap A_k$ is stationary. (Note: for all cubs $C \subseteq \omega_1$, $\{s \colon s = \sum_{i<\omega}(1 + \eta + \sum_{\beta<\alpha} \tau_\beta^i)$ and $\alpha \in C\}$ is a cub in M.) Let

$$\psi(x, y) \equiv x = y.$$

We can finish the proof as in Theorem 3.

We shall show in the following that $\mathrm{Ba}(aa)$ is undecidable, thus answering a question of Eklof and Mekler [4]. We set

$$\chi(x) \equiv Q_1 y(y \leqslant x) \wedge \forall y(y \leqslant x \to \neg Q_1 z(z \leqslant y) \vee \neg Q_1 z(z \leqslant y \cap x)).$$

Let $\mathfrak{B}$ be a Boolean algebra, $a \in |\mathfrak{B}|$. If $\mathfrak{B} \models \chi[a]$, a is called $\aleph_1$-*atom*. For any $a \in |\mathfrak{B}|$, $\mathfrak{B}[a]$ denotes the Boolean algebra consisting of all elements of $\mathfrak{B}$ which are less than or equal to a. Let $\mathfrak{M}$ be a linear ordering with first element. $\mathfrak{J}(\mathfrak{M})$ denotes the Boolean algebra generated by all left-closed right-open intervals of $\mathfrak{M}$. Let

$$\rho(s, x, y) \equiv \chi(x) \wedge \forall z(s(z) \wedge z < x \wedge \neg\chi(z) \to z \leqslant y)$$
$$\wedge \forall z(\forall u(s(u) \wedge u < x \wedge \neg\chi(u) \to u \leqslant z) \to y \leqslant z).$$

That means, if $\mathfrak{M} \models \rho[s, a, b]$, then a is an $\aleph_1$-atom and b is the supremum of all elements of s which are less than a, and are not $\aleph_1$-atoms.

THEOREM 6. $\mathrm{Ba}(aa)$ *is undecidable.*

PROOF. Let $\mathfrak{G}$ and A_k $(k < \omega)$ be as in the proof of Theorem 3. Let $\mathfrak{B} = \mathfrak{J}(\sum_{i<\omega} \Phi(A_i))$ and let a_i be the least element of $\Phi(A_{i+1})$. So $\mathfrak{B} \models \chi[b]$ iff $b = [a, a_i) \cup b_1 \cup \cdots \cup b_n$ where each b_k is a countable interval and $a_{i-1} \leqslant a < a_i$ (where a_{-1} is the least element of $\Phi(A_0)$). Let

$$\varphi_0(x) \equiv \chi(x); \quad \varphi_1(x, y) \equiv \text{stat } s \exists uv(\rho(s, x, u) \wedge \rho(s, y, v)).$$

If $b = [a, a_i) \cup b_1 \cup \cdots \cup b_n$ and $b' = [a', a_j) \cup b'_1 \cup \cdots \cup b'_m$ are $\aleph_1$-atoms, then $\mathfrak{B} \models \varphi_1[b, b']$ iff $A_i \cap A_j$ is stationary. Let

$$\psi(x, y) \equiv \chi(x) \wedge \chi(y) \wedge \chi(x \cap y).$$

With b, b' as above, $\mathfrak{B} \models \psi[b, b']$ iff $a_i = a_j$. Thus it is immediately seen that

$$\langle \varphi_0^{\mathfrak{B}}, \varphi_1^{\mathfrak{B}} \rangle / \psi^{\mathfrak{B}} \cong \mathfrak{G}.$$

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