UNDECIDABLE THEORIES IN STATIONARY LOGIC

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Abstract. It is shown that the theories of the following classes become undecidable in stationary logic: Well orderings with one unary predicate, linear orderings, and Boolean algebras. This is done by interpreting the theory of symmetric reflexive graphs.

1. Introduction. This paper establishes that the $L(aa)$-theories of well orderings with a distinguished subset (WoP), linear orderings (Lin) and Boolean algebras (Ba) are all undecidable. This result is of interest since some decidability results (e.g., Abelian groups, reals [4], and ordinals [7] extend from $L$ to $L(Q)$ and then to $L(aa)$). It is known that the $L(Q)$-theories of Lin, Ba [2] and WoP (can be shown similar to the decidability of the $L(Q)$-theory of Lin) are decidable. Also the $L(aa)$-theory of $\omega_1$ with a distinguished subset is decidable. This is a special case of the decidability of the monadic theory of $\omega_1$ [3].

2. Definitions and notations. We shall implicitly assume throughout the paper that the structures we consider are uncountable. For a model $\mathfrak{M}$ for $L$ its universe is denoted by $|\mathfrak{M}|$. Let $P_1(\mathfrak{M})$ denote the set of all countable subsets of $|\mathfrak{M}|$. A set $X \subseteq P_1(\mathfrak{M})$ is unbounded, if every $s_0 \in P_1(\mathfrak{M})$ is a subset of some $s \in X$. $X$ is closed, if the union of each increasing sequence $s_0 \subseteq s_1 \subseteq \cdots \subseteq s_n \subseteq \cdots$ of elements of $X$ is again an element of $X$. A set $X$ is cub, if it is closed and unbounded. $\mathcal{F}(\mathfrak{M})$ denotes the filter on $P_1(\mathfrak{M})$ generated by all cub sets. We define

$$\mathfrak{M} \vdash aa \varphi(s) \quad \text{iff} \quad \{ s \in P_1(\mathfrak{M}) : \mathfrak{M} \vdash \varphi(s) \} \in \mathcal{F}(\mathfrak{M}).$$

If $\mathfrak{M} \vdash aa \varphi(s)$, we say that property $\varphi(s)$ holds for almost all countable $s$.

We assume that the reader is familiar with [1]. In that paper it is shown that the quantifier $Q_1$, read “there exist uncountably many”, is definable in $L(aa)$ by the formula

$$\text{stat } s \exists x (\varphi(x) \land \neg s(x)).$$

Thus $L(Q_1)$ is a sublogic of $L(aa)$.

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Let $T$ be a theory, $\mathcal{M} \models T$ and $\varphi(x_1, \ldots, x_n)$, $\psi(s)$ formulas of the corresponding $L(aa)$-language for $T$. Then we set

$$\varphi^\mathcal{M} = \{ \bar{a} \in |\mathcal{M}| : \mathcal{M} \models \varphi(\bar{a}) \} ;$$

$$\psi^\mathcal{M} = \{ s \in P_1(|\mathcal{M}|) : \mathcal{M} \models \psi(s) \} .$$

If $T$ is the elementary theory of some class of structures, then $T(aa)$ denotes the corresponding $L(aa)$-theory of the same class of models.

3. Preliminaries. Let $\text{Gr}$ be the theory of infinite symmetric reflexive graphs. In order to show the undecidability of any theory $T$ it is enough to show that $\text{Gr}$ is interpretable in $T$, i.e. there are formulas $\varphi_0(x)$, $\varphi_1(x, y)$ and $\psi(x, y)$ such that

for each countable symmetric graph $\mathcal{G} = \langle \omega, R \rangle$ there is

$$\text{(\ast)}$$

a model $\mathcal{M} \models T$ such that $\psi^\mathcal{M}$ is a congruence relation on

$\langle \varphi_0^\mathcal{M}, \varphi_1^\mathcal{M} \rangle$ and $\langle \varphi_0^\mathcal{M}, \varphi_1^\mathcal{M} \rangle / \psi^\mathcal{M} \cong \mathcal{G}.$

Here we use the fact that the theory of infinite symmetric reflexive graphs is hereditarily undecidable. Let $T^*$ be $T$ together with a sentence $\rho$ which says of a model $\mathcal{M} \models T$, that $(\mathcal{M}, \psi^\mathcal{M})/\psi^\mathcal{M}$ is a symmetric reflexive graph. So $T^*$ and hence $T$ is undecidable.

A proof of the following well-known theorem can be found in [6]:

**Theorem 1 (Ulam).** There is a set $\{ S_\alpha : \alpha < \omega_1 \}$ of pairwise disjoint stationary subsets of $\omega_1$.

4. Undecidability proofs. We set

$$\sigma(s, x, y) \equiv \forall z(s(z) \land z < x \rightarrow z \land y)$$

$$\land \forall z(z < y \rightarrow \exists u(s(u) \land z < u \land u < y \land u < x)).$$

Thus, if $\mathcal{M}$ is an ordering, $s \subseteq |\mathcal{M}|$ and $a, b \in |\mathcal{M}|$, and $\mathcal{M} \models \sigma(s, a, b)$, then $b = \sup \{ c \in |\mathcal{M}| : s(c) \land c < a \}$.

**Lemma 2.** Let $A, B \subseteq \omega_1$, $\alpha, \beta, \gamma$ ordinals with $\alpha \neq \beta$, $\omega_1(\alpha + 1), \omega_1(\beta + 1) < \gamma$, $P = \{ \omega_1 \cdot \alpha + \delta : \delta \in A \} \cup \{ \omega_1 \cdot \beta + \delta : \delta \in B \}$, $\mathcal{M} = \langle \gamma, <, P \rangle$. Then

$$\text{stat } s \exists xy(\sigma(s, \omega_1(\alpha + 1), x) \land \sigma(s, \omega_1(\beta + 1), y) \land P(x) \land P(y))$$

iff $A \cap B$ is stationary.

**Proof.** For $\mathcal{X} \in P_1(|\mathcal{M}|)$ let $\mathcal{X}^* = \{ s \in \mathcal{X} : \text{ for all } \delta < \omega_1, \omega_1 \cdot \alpha + \delta \in s \text{ iff } \omega_1 \cdot \beta + \delta \in s \}$. Let

$$\varphi(s) \equiv \exists xy(\sigma(s, \omega_1(\alpha + 1), x) \land \sigma(s, \omega_1(\beta + 1), y) \land P(x) \land P(y)).$$

Assume $\mathcal{M} \models \text{stat } s \varphi(s)$. Let $C \subseteq \omega_1$. We set

$$\mathcal{X}_C = \left\{ s \in P_1(|\mathcal{M}|) : \text{ there is } \delta \in C \text{ with } \sigma(s, \omega_1(\alpha + 1), \omega_1 \cdot \alpha + \delta) \text{ and } \sigma(s, \omega_1(\beta + 1), \omega_1 \cdot \beta + \delta) \right\}.$$
If $C$ is closed unbounded, then $\mathcal{F}_C \in F(\mathcal{M})$. Assume $A \cap B$ is not stationary. Then there is a closed unbounded subset $C$ of $\omega_1$ with $C \cap (A \cap B) = \emptyset$. Then $\mathcal{F}_C \in F(\mathcal{M})$ but for no $s \in \mathcal{F}_C$, $\mathcal{M} \models \varphi(s)$. Thus $\mathcal{M} \models \text{stat } \varphi(s)$. Assume on the other hand that $A \cap B$ is stationary. Let $\mathcal{X} \in F(\mathcal{M})$. We set

$$\mathcal{X} = \{ s \in \mathcal{X} : \text{there is } \delta < \omega_1 \text{ with } \sigma(s, \omega_1(\alpha + 1), \omega_1 \cdot \alpha + \delta) \text{ and } \sigma(s, \omega_1(\beta + 1), \omega_1 \cdot \beta + \delta) \}.$$ 

Then $\mathcal{X} \in F(\mathcal{M})$. We set

$$X = \{ s < \omega_1 : \text{there is } s \in \mathcal{X} \text{ with } \sigma(s, \omega_1(\alpha + 1), \omega_1 \cdot \alpha + \delta) \}.$$ 

Then $X$ is a closed unbounded subset of $\omega_1$ and thus there is $\delta \in X \cap A \cap B$. But then there is $s \in \mathcal{X} \cap \mathcal{M} \models \varphi(s)$ and thus $\mathcal{M} \models \text{stat } \varphi(s)$. □

**Theorem 3.** WoP($\alpha \alpha$) is undecidable.

**Proof.** We set

$$\varphi_0(x) = \exists y(y < x) \land \forall z(z < x \rightarrow Q_z(z < u < x));$$

$$\varphi_1(x, y) \equiv \varphi_0(x) \land \varphi_0(y) \land \text{stat } s \exists z u(\sigma(s, x, z) \land \sigma(s, y, u) \land P(z) \land P(u));$$

$$\psi(x, y) \equiv x = y.$$ 

Let $\Theta = (\omega, R)$ be a countable symmetric reflexive graph. Let $\{S_n : n < \omega\}$ be a set of pairwise disjoint subsets of $\omega_1$ and let $h : \omega \times \omega \rightarrow \omega$ be a 1-1 function. For $k < \omega$ let

$$A_k = \bigcup \{ S_{h(i,j)} : (k, i) \in R \} \cup \bigcup \{ S_{h(i,j)} : (k, i) \in R \}.$$ 

Then each $A_k$ is a stationary set and $A_k \cap A_i$ is stationary iff $(k, i) \in R$. We set

$$P = \bigcup_{k < \omega} \{ \omega_1 : k + \alpha : A \in A_k \}, \quad \mathcal{M} = \langle \omega_1, \cdot, \omega, <, P \rangle.$$ 

Then Lemma 2 implies immediately that

$$\langle \varphi_0^{\mathcal{M}}, \varphi_1^{\mathcal{M}} \rangle \models \Theta.$$ 

$\mathcal{M}$ is an $\mathcal{N}_1$-like dense linear ordering if $\text{card}(\mathcal{M}) = \mathcal{N}_1$ and for each $a \in \mathcal{M}$, the set $\{ b \in \mathcal{M} : b < a \}$ forms an order which is isomorphic to $\eta$ or $1 + \eta$. For each $A \subseteq \omega_1$ let $\Phi(A)$ be the $\mathcal{N}_1$-like dense linear ordering $1 + \eta + \sum_{a < \omega_1} \tau_a$ where

$$\tau_a = \begin{cases} 1 + \eta & \text{for } a \in A; \\ \eta & \text{otherwise.} \end{cases}$$ 

A proof of the following lemma can be found in [5]:

**Lemma 4 (Conway).** For each $\mathcal{N}_1$-like dense linear ordering $\mathcal{M}$ with first element there exists $A \subseteq \omega_1$ with $\Phi(A) \approx \mathcal{M}$. For $A, B \subseteq \omega_1$, $\Phi(A) \approx \Phi(B)$ iff there is some closed unbounded set $C \subseteq \omega_1$ with $A \cap C = B \cap C$.

We use this lemma to show

**Theorem 5.** Lin($\alpha \alpha$) is undecidable.
Proof. Let $\emptyset$ and $A_k (k < \omega)$ be as in the proof of Theorem 3. Let $\mathcal{M} = \Sigma_{i<\omega} \Phi(A_i)$ and let $a_i$ be the least element of $\Phi(A_{i+1})$. Let 

$$\varphi_0(x) \equiv \exists y (y < x) \land \forall y (y < x \rightarrow Q_1 z (y < z \land z < x)).$$

Then $\varphi_0^{\mathcal{M}} = \{a_i; i < \omega\}$. Let

$$\varphi_1(x, y) \equiv \text{stat } s \exists w z (\sigma(s, x, w) \land \sigma(s, y, z)).$$

As in Lemma 2 we have $\mathcal{M} \vdash \varphi_1[a_i, a_k]$ iff $A_i \cap A_k$ is stationary. (Note: for all cubs $C \subseteq \omega_1$, $\{s: s = \Sigma_{i<\omega} (1 + \eta + \Sigma_{\beta<\alpha} \eta^\beta) \text{ and } \alpha \in C\}$ is a cub in $M$.) Let

$$\psi(x, y) \equiv x = y.$$

We can finish the proof as in Theorem 3.

We shall show in the following that $Ba(aa)$ is undecidable, thus answering a question of Eklof and Mekler [4]. We set

$$\chi(x) \equiv Q_1 y (y < x) \land \forall y (y < x \rightarrow \neg Q_1 z (z < y) \lor \neg Q_1 z (z < y \land x)).$$

Let $\mathfrak{B}$ be a Boolean algebra, $a \in |\mathfrak{B}|$. If $\mathfrak{B} \models \chi[a]$, $a$ is called $\mathfrak{N}_1$-atom. For any $a \in |\mathfrak{B}|$, $\mathfrak{B}[a]$ denotes the Boolean algebra consisting of all elements of $\mathfrak{B}$ which are less than or equal to $a$. Let $\mathfrak{M}$ be a linear ordering with first element. $\mathfrak{Y}(\mathcal{M})$ denotes the Boolean algebra generated by all left-closed right-open intervals of $\mathcal{M}$. Let

$$\rho(s, x, y) \equiv \chi(x) \land \forall z (z < x \land \neg \chi(z) \rightarrow z < y)$$

$$\land \forall z (\forall u (s(u) \land u < x \land \neg \chi(u) \rightarrow u < z) \rightarrow y < z).$$

That means, if $\mathfrak{M} \models \rho[s, a, b]$, then $a$ is an $\mathfrak{N}_1$-atom and $b$ is the supremum of all elements of $s$ which are less than $a$, and are not $\mathfrak{N}_1$-atoms.

Theorem 6. $Ba(aa)$ is undecidable.

Proof. Let $\emptyset$ and $A_k (k < \omega)$ be as in the proof of Theorem 3. Let $\mathfrak{B} = \mathfrak{Y}(\Sigma_{i<\omega} \Phi(A_i))$ and let $a_i$ be the least element of $\Phi(A_{i+1})$. So $\mathfrak{B} \models \chi[b]$ iff $b = [a, a_i] \cup b_1 \cup \cdots \cup b_n$ where each $b_k$ is a countable interval and $a_i-1 \leq a < a_i$ (where $a_i$ is the least element of $\Phi(A_0\emptyset)$). Let

$$\varphi_0(x) \equiv \chi(x); \ \ \ \varphi_1(x, y) \equiv \text{stat } s \exists w v (\rho(s, x, u) \land \rho(s, y, v)).$$

If $b = [a, a_i] \cup b_1 \cup \cdots \cup b_n$ and $b' = [a', a_j] \cup b'_1 \cup \cdots \cup b'_m$ are $\mathfrak{N}_1$-atoms, then $\mathfrak{B} \models \varphi_1[b, b']$ iff $A_i \cap A_j$ is stationary. Let

$$\psi(x, y) \equiv \chi(x) \land \chi(y) \land \chi(x \land y).$$

With $b, b'$ as above, $\mathfrak{B} \models \psi[b, b']$ iff $a_i = a_j$. Thus it is immediately seen that

$$\langle \varphi_0^{\mathfrak{B}}, \varphi_1^{\mathfrak{B}} \rangle \models \psi^{\mathfrak{B}} \equiv \emptyset.$$

References


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