THE NONHOMOGENEITY OF THE \( E \)-TREE—ANSWER TO A PROBLEM RAISED BY D. JENSEN AND A. EHRENFEUCHT

DIDIER MISERCQUE

Abstract. We prove that the ordered system of all \( C^1EP \)'s, under the order “admits embedding in” is not homogeneous. This answers a problem raised in [2].

1. Introduction. We assume familiarity with [2]. \( L \) denotes the lattice of all \( \forall_1 \)-sentences of Peano Arithmetic (PA) modulo PA. \( \alpha, \beta, \gamma \ldots \) denote elements of \( L \) (we often identify \( \forall_1 \)-sentences with their equivalence classes). 0 and 1 denote respectively the minimum element and the maximum element of \( L \).

By the \( E \)-tree we mean, the class of all prime filters of \( L \) under the partial ordering of reverse inclusion \( \supset \). By a \( C^1EP \) is meant the set of all existential sentences (without parameters) satisfied in some model of PA. The following results are well known (see [2] and [4]).

Lemma 1. F is a prime filter of \( L \) iff
\[ \neg (L \setminus F) \text{ is a } C^1EP. \]
This gives an isomorphism between the \( E \)-tree and the ordered system of all \( C^1EP \)'s.

Lemma 1.2. (i) The set of the predecessors of an element of the \( E \)-tree is totally ordered.
(ii) The \( E \)-tree has a minimum element (i.e. \( L \setminus \{0\} \)) and each of its branches has a maximal element.

Jensen and Ehrenfeucht ask [2, p. 243] whether the \( E \)-tree is homogeneous in the sense that any pair of nonminimal, nonmaximal elements can be exchanged by an automorphism.

2. Preliminary results.

Lemma 2.1. The \( E \)-tree has an element \( F \) such that
(i) \( F \) is not maximal,
(ii) \( F \) is not minimal,
(iii) \( F \) has no immediate predecessor,
(iv) if \( B \) is any branch of the \( E \)-tree containing \( F \), then \( F \) has an immediate successor in \( B \).

Received by the editors January 21, 1981.


1Supported in part by an IRSIA grant. I am grateful to the referee for helpful suggestions.

© 1982 American Mathematical Society

0002-9939/81/0000-1091/$01.75

573
Proof. Let $\theta$ be a $\forall_1$-sentence independent of $\text{PA}$ such that $\text{PA} + \neg \theta$ and $\text{PA}$ have the same $\forall_1$-theorems; (such a formula exists by a result of Kreisel, cf. §1 of [1]). We denote by $E_\theta$ the class of all prime filters of $L$ containing $\theta$ and ordered by $\supseteq$. It is easily shown that each branch of $E_\theta$ has a maximum element. Therefore $E_\theta$ has (at least) one maximal element $F_\theta$. We will show that $F_\theta$ has the required properties.

(i) Let $I = L \setminus F_\theta$, we have $\theta \notin I$. Denote by $T$ the theory $\text{PA} + \neg I + \neg \theta$. $T$ is consistent because if $\text{PA} + \neg I \vdash \theta$, then

$$\exists \varphi \in I \quad \text{PA} + \neg \varphi \vdash \theta,$$

and thus

$$\text{PA} \vdash \varphi.$$}

This is impossible because $\text{PA} + \neg \varphi$ is consistent. Obviously, the prime filter of all $\forall_1$-sentences true in any model of $T$ is properly contained in $F_\theta$. So $F_\theta$ is not maximal.

(ii) If $F_\theta = L \setminus \{0\}$, then $E_\theta = \{L \setminus \{0\}\}$ and the only prime filter of $L$ containing $\theta$ is $\forall_1(N)$. We infer that $\text{PA} + \theta \equiv \text{PA} + (L \setminus \{0\})$. This is impossible because $\text{PA} + \theta$ is an R.E. theory and $\text{PA} + (L \setminus \{0\})$ is a $\forall_1^0$-non-R.E. theory.

(iii) Suppose that $F_\theta$ has an immediate predecessor $F'$. Then

$$\forall \beta \in L \setminus F_\theta \quad \text{PA} + F_\theta + \beta \vdash F'$$

(because, if this theory is consistent, the class of all $\forall_1$-sentences true in any model of $T$ is a prime filter of $L$ containing properly $F_\theta$ and therefore $F'$; if $\text{PA} + F_\theta + \beta$ is not consistent, the result is obvious). Therefore

$$\forall \alpha \in F' \setminus F_\theta \quad \forall \beta \in L \setminus F_\theta \quad \exists \gamma \in F_\theta \quad \text{PA} + \gamma + \beta \vdash \alpha,$$

or

$$(*) \quad \text{PA} + \neg \alpha + \beta \vdash \neg \gamma.$$}

We also have that $F_\theta$ is a maximal element of $E_\theta n$ and therefore $\forall \xi \in F_\theta \quad \text{PA} + \neg (L \setminus F_\theta) + \theta + \neg \xi$ is an inconsistent theory (because, if this theory is consistent, the prime filter of all $\forall_1$-sentences true in any model of this theory is an element of $E_\theta$ properly contained in $F_\theta$). Therefore,

$$(**) \quad \forall \xi \in F_\theta \quad \exists \rho \in L \setminus F_\theta \quad \text{PA} + \theta \vdash \xi \lor \rho.$$}

Let $\alpha \in F' \setminus F_\theta$. By a result of Solovary (cf. [1, Theorem 2.7]), we know that there is a $\forall_1$-sentence $\varphi$, independent of $\text{PA} + \theta + \neg \alpha$, such that

(I) $\text{PA} + \theta + \neg \alpha + \varphi$ and $\text{PA} + \theta + \neg \alpha$ have the same $\exists_1$-theorems,

(II) $\text{PA} + \theta + \neg \alpha + \neg \varphi$ and $\text{PA} + \theta + \neg \alpha$ have the same $\forall_1$-theorems. $\varphi \notin F_\theta$; because, if $\varphi \in F_\theta$ then, by $(**)$, we have

$$\exists \varphi' \in L \setminus F_\theta \quad \text{PA} + \theta \vdash \varphi \lor \varphi',$$

$$\text{PA} + \theta + \neg \alpha \lor \varphi \lor \varphi',$$

$$\text{PA} + \theta + \neg \alpha + \neg \varphi \lor \varphi',$$

$$\text{PA} + \theta + \neg \alpha \lor \varphi \lor \varphi'$$}
and

\[ PA \vdash \theta \Rightarrow \alpha \lor \varphi', \]

but \( \theta \in F_\theta, \alpha \lor \varphi' \notin F_\theta \) and \( \theta \leq \alpha \lor \varphi' \). Contradiction!

If \( \varphi \in L \setminus F_\theta \), we have by (*)

\[ \exists \gamma \in F_\theta \quad PA + \neg \alpha + \varphi \vdash \neg \gamma, \]

\[ PA + \theta + \neg \alpha + \varphi \vdash \neg \gamma, \]

\[ PA + \theta + \neg \alpha \vdash \neg \gamma, \]

and

\[ PA \vdash \theta \land \gamma \Rightarrow \alpha, \]

but \( \alpha \notin F_\theta, \theta \land \gamma \in F_\theta \) and \( \theta \land \gamma \leq \alpha \). Contradiction! \( F_\theta \) has therefore no immediate predecessor.

(iv) Let \( B \) be a branch of the \( E \)-tree containing \( F_\theta \). \( F_\theta \) is a maximal element of \( E_\theta \), and therefore \( F_\theta \) is the greatest element of \( B \) containing \( \theta \). Let \( A = \{ F \in B \mid \theta \notin F \} \).

It is straightforward to check that \( F' = \bigcup_{F \in A} F \) is the lowest element of \( B \) which does not contain \( \theta \). \( F' \) is, of course, an immediate successor of \( F_\theta \).

**Lemma 2.2.** If \( F \) is any maximal element of the \( E \)-tree, then \( F \) has no immediate predecessor.

**Proof.** We use the same kind of argument as in the proof of Lemma 2.1(iii). Let \( \theta = 1 \). (We delete, of course, the requirement "\( PA + \neg \theta \) and \( PA \) have the same \( \forall \)-theorems" which is not used in the proof of Lemma 2.1(iii).) Now \( F_\theta \) becomes any maximal element of the \( E \)-tree.

3. The main result.

**Theorem 3.1.** The \( E \)-tree is not homogeneous.

**Proof.** This is an immediate consequence of Lemmas 2.1 and 2.2, for, in the notation of Lemma 2.1, \( F_\theta \) and its immediate successor \( F' \) are neither minimal nor maximal and yet cannot be exchanged by an automorphism of the \( E \)-tree.

**References**


**Department of Mathematics, University of Brussels, 1050 Brussels, Belgium**