PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 4, April 1982

THE NONHOMOGENEITY OF THE E-TREE—ANSWER TO A PROBLEM RAISED BY D. JENSEN AND A. EHRENFEUCHT

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ABSTRACT. We prove that the ordered system of all C^1EP's, under the order "admits embedding in" is not homogeneous. This answers a problem raised in [2].

1. Introduction. We assume familiarity with [2]. L denotes the lattice of all $\forall_1$-sentences of Peano Arithmetic (PA) modulo PA. $\alpha, \beta, \gamma, \ldots$ denote elements of L (we often identify $\forall_1$-sentences with their equivalence classes). 0 and 1 denote respectively the minimum element and the maximum element of L.

By the E-tree we mean, the class of all prime filters of L under the partial ordering of reverse inclusion $\supset$. By a C^1EP is meant the set of all existential sentences (without parameters) satisfied in some model of PA. The following results are well known (see [2] and [4]).

LEMMA 1.1. *F is a prime filter of L iff*

$$\neg(L \setminus F) \text{ is a } C^1EP.$$

This gives an isomorphism between the E-tree and the ordered system of all C^1EP's.

LEMMA 1.2. (i) *The set of the predecessors of an element of the E-tree is totally ordered.*

(ii) *The E-tree has a minimum element (i.e. $L \setminus \{0\}$) and each of its branches has a maximal element.*

Jensen and Ehrenfeucht ask [2, p. 243] whether the E-tree is homogeneous in the sense that any pair of nonminimal, nonmaximal elements can be exchanged by an automorphism.

2. Preliminary results.

LEMMA 2.1. *The E-tree has an element F such that*

(i) *F is not maximal,*

(ii) *F is not minimal,*

(iii) *F has no immediate predecessor,*

(iv) *if B is any branch of the E-tree containing F, then F has an immediate successor in B.*

Received by the editors January 21, 1981.

AMS (MOS) subject classifications (1970). Primary 02H20.

[1]Supported in part by an IRSIA grant. I am grateful to the referee for helpful suggestions.

PROOF. Let θ be a $\forall_1$-sentence independent of PA such that PA $+ \neg\theta$ and PA have the same $\forall_1$-theorems; (such a formula exists by a result of Kreisel, cf. §1 of [**1**]). We denote by E_θ the class of all prime filters of L containing θ and ordered by $\supset$. It is easily shown that each branch of E_θ has a maximum element. Therefore E_θ has (at least) one maximal element F_θ. We will show that F_θ has the required properties.

(i) Let $I = L \setminus F_\theta$, we have $\theta \notin I$. Denote by T the theory PA $+ \neg I + \neg\theta$. T is consistent because if PA $+ \neg I \vdash \theta$, then

$$\exists \varphi \in I \qquad \begin{aligned} &\mathrm{PA} + \neg\varphi \vdash \theta, \\ &\mathrm{PA} + \neg\theta \vdash \varphi, \end{aligned}$$

and thus

$$\mathrm{PA} \vdash \varphi.$$

This is impossible because PA $+ \neg\varphi$ is consistent. Obviously, the prime filter of all $\forall_1$-sentences true in any model of T is properly contained in F_θ. So F_θ is not maximal.

(ii) If $F_\theta = L \setminus \{0\}$, then $E_\theta = \{L \setminus \{0\}\}$ and the only prime filter of L containing θ is $\forall_1(\mathbf{N})$. We infer that PA $+ \theta \equiv$ PA $+ (L \setminus \{0\})$. This is impossible because PA $+ \theta$ is an R.E. theory and PA $+ (L \setminus \{0\})$ is a π_1^0-non-R.E. theory.

(iii) Suppose that F_θ has an immediate predecessor F'. Then

$$\forall \beta \in L \setminus F_\theta \qquad \mathrm{PA} + F_\theta + \beta \vdash F'$$

(because, if this theory is consistent, the class of all $\forall_1$-sentences true in any model of T is a prime filter of L containing properly F_θ and therefore F'; if PA $+ F_\theta + \beta$ is not consistent, the result is obvious). Therefore

$$\forall \alpha \in F' \setminus F_\theta \qquad \forall \beta \in L \setminus F_\theta \qquad \exists \gamma \in F_\theta \qquad \mathrm{PA} + \gamma + \beta \vdash \alpha,$$

or

$$(*) \qquad \mathrm{PA} + \neg\alpha + \beta \vdash \neg\gamma.$$

We also have that F_θ is a maximal element of E_θn and therefore $\forall \xi \in F_\theta$ PA $+ \neg(L \setminus F_\theta) + \theta + \neg\xi$ is an inconsistent theory (because, if this theory is consistent, the prime filter of all $\forall_1$-sentences true in any model of this theory is an element of E_θ properly contained in F_θ). Therefore,

$$(**) \qquad \forall \xi \in F_\theta \qquad \exists \rho \in L \setminus F_\theta \qquad \mathrm{PA} + \theta \vdash \xi \vee \rho.$$

Let $\alpha \in F' \setminus F_\theta$. By a result of Solovary (cf. [**1**, Theorem 2.7]), we know that there is a $\forall_1$-sentence φ, independent of PA $+ \theta + \neg\alpha$, such that

(I) PA $+ \theta + \neg\alpha + \varphi$ and PA $+ \theta + \neg\alpha$ have the same $\exists_1$-theorems,

(II) PA $+ \theta + \neg\alpha + \neg\varphi$ and PA $+ \theta + \neg\alpha$ have the same $\forall_1$-theorems. $\varphi \notin F_\theta$; because, if $\varphi \in F_\theta$ then, by $(**)$, we have

$$\exists \varphi' \in L \setminus F_\theta \qquad \begin{aligned} &\mathrm{PA} + \theta \vdash \varphi \vee \varphi', \\ &\mathrm{PA} + \theta + \neg\alpha \vdash \varphi \vee \varphi', \\ &\mathrm{PA} + \theta + \neg\alpha + \neg\varphi \vdash \varphi', \\ &\mathrm{PA} + \theta + \neg\alpha \vdash \varphi', \end{aligned}$$

and

$$\mathrm{PA} \vdash \theta \Rightarrow \alpha \vee \varphi',$$

but $\theta \in F_\theta$, $\alpha \vee \varphi' \notin F_\theta$ and $\theta \leqslant \alpha \vee \varphi'$. Contradiction!

If $\varphi \in L \setminus F_\theta$, we have by $(*)$

$$\begin{aligned} \exists \gamma \in F_\theta \quad & \mathrm{PA} + \neg\alpha + \varphi \vdash \neg\gamma, \\ & \mathrm{PA} + \theta + \neg\alpha + \varphi \vdash \neg\gamma, \\ & \mathrm{PA} + \theta + \neg\alpha \vdash \neg\gamma, \end{aligned}$$

and

$$\mathrm{PA} \vdash \theta \wedge \gamma \Rightarrow \alpha,$$

but $\alpha \notin F_\theta$, $\theta \wedge \gamma \in F_\theta$ and $\theta \wedge \gamma \leqslant \alpha$. Contradiction! F_θ has therefore no immediate predecessor.

(iv) Let B be a branch of the E-tree containing F_θ. F_θ is a maximal element of E_θ, and therefore F_θ is the greatest element of B containing θ. Let $A = \{F \in B \mid \theta \notin F\}$. It is straightforward to check that $F' = \bigcup_{F \in A} F$ is the lowest element of B which does not contain θ. F' is, of course, an immediate successor of F_θ.

LEMMA 2.2. *If F is any maximal element of the E-tree, then F has no immediate predecessor.*

PROOF. We use the same kind of argument as in the proof of Lemma 2.1(iii). Let $\theta = 1$. (We delete, of course, the requirement "PA + $\neg\theta$ and PA have the same $\forall_1$-theorems" which is not used in the proof of Lemma 2.1(iii).) Now F_θ becomes any maximal element of the E-tree.

3. The main result.

THEOREM 3.1. *The E-tree is not homogeneous.*

PROOF. This is an immediate consequence of Lemmas 2.1 and 2.2, for, in the notation of Lemma 2.1, F_θ and its immediate successor F' are neither minimal nor maximal and yet cannot be exchanged by an automorphism of the E-tree.

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