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A SYMPLECTIC FIXED POINT THEOREM ON OPEN MANIFOLDS

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ABSTRACT. In 1968 Bourgin proved that every measure-preserving, orientation-preserving homeomorphism of the open disk has a fixed point, and he asked whether such a result held in higher dimensions. Asimov, in 1976, constructed counterexamples in all higher dimensions. In this paper we answer a weakened form of Bourgin's question dealing with symplectic diffeomorphisms: every symplectic diffeomorphism of an even-dimensional cell sufficiently close to the identity in the C^1-fine topology has a fixed point. This result follows from a more general result on open manifolds and symplectic diffeomorphisms.

Introduction. Fixed point theorems for area-preserving mappings have a history which dates back to Poincaré's "last geometric theorem", i.e., any area-preserving mapping of an annulus which twists the boundary curves in opposite directions has at least two fixed points. More recently it has been proved that any area-preserving, orientation-preserving mapping of the two-dimensional sphere into itself possesses at least two distinct fixed points (see [**N**, **Si**]). In the setting of noncompact manifolds, Bourgin [**B**] showed that any measure-preserving, orientation-preserving homeomorphism of the open two-cell B^2 has a fixed point. For Bourgin's theorem one assumes that the measure is finite on B^2 and that the measure of a nonempty open set is positive. Bourgin also gave a counterexample to the generalization of the theorem for the open ball in $\mathbf{R}^{135}$ and asked the question whether his theorem remains valid for the open balls in low dimensions. In [**As**] Asimov constructed counterexamples for all dimensions greater than two and actually got a flow of measure-preserving, orientation-preserving diffeomorphisms with no periodic points.

To formulate our results and place the comments above into our framework, we need some concepts from symplectic geometry. A smooth manifold is called *symplectic* if there exists a nondegenerate, closed, differentiable 2-form ω defined on M. A differentiable mapping f of M into itself is called *symplectic* if f preserves the form ω. We refer to the texts by Abraham and Marsden [**A & M**] and Arnold [**A**] for the general background in symplectic geometry.

We reformulate Bourgin's question to ask: does every symplectic mapping of a $2n$-dimensional cell, equipped with a symplectic structure, have a fixed point? Using a generalization of a theorem of Weinstein [$\mathbf{W_2}$], we answer this question affirmatively for mappings sufficiently close to the identity.

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1. Preliminaries. All manifolds are assumed to be finite-dimensional, C^∞-smooth, and without boundary. A manifold M is *open* if M has no compact components. Let $\varepsilon(M)$ denote the ends of M, and let $\tilde{M} = M \cup \varepsilon(M)$ be the completion of M. We consider manifolds M where the number of ends, denoted by $e(M)$, is finite and where $\tilde{M}$ has a smooth manifold structure without boundary. For the general problem of completing an open manifold with finitely many ends see Siebenmann's thesis [**S**].

If M is a manifold with symplectic form ω, then $\operatorname{Diff}(M, \omega)$ denotes the group of symplectic diffeomorphisms of M. The closed one-forms on M are denoted by $Z^1(M)$. Both of these function spaces are topologized with the C^1-fine topology. See [**H**, p. 35] for a good account of the C^1-fine topology.

We require the basic formalism of "cotangent co-ordinates" contained in the following theorem of Weinstein.

THEOREM 1.1 [$\mathbf{W_1}$, Proposition (2.7.4) or $\mathbf{W_2}$, Theorem 7.2]. *If (M, ω) is a symplectic manifold, then there is a C^1-fine neighborhood $A \subset \operatorname{Diff}(M, \omega)$ containing the identity map, a C^1-fine neighborhood $B \subset Z^1(M)$ containing the zero form, and a homeomorphism $V\colon A \to B$. If $f \in A$, then a point $x \in M$ is a fixed point of f if and only if $(V(f))(x) = 0$.*

PROOF. If f is in $\operatorname{Diff}(M, \omega)$, then the graph of f is a Lagrangian submanifold of $M \times M$ with the symplectic structure $\pi_1^*\omega - \pi_2^*\omega$, where π_1 and π_2 are the projections. There exists a neighborhood U of the diagonal $\Delta(M) = \{(m, m)\colon m \in M\}$ and a bijection of U onto a neighborhood W of the zero-section in T^*M, taking Lagrangian submanifolds of U onto Lagrangian submanifolds lying in W. If f is close enough to the identity, in the sense that the graph of f is contained in U, then there is a one-form $V(f) \in Z^1(M)$ whose image is contained in W. Clearly, $f(x) = x$ if and only if $(V(f))(x) = 0$. $\square$

Various fixed point theorems in symplectic geometry result from Theorem 1.1. For examples see [**M**, **N**, **S**, $\mathbf{W_1}$, and $\mathbf{W_2}$]. Let M be a compact manifold and η a closed one-form. Define $c(\eta)$ to be the number of zeros of η. Define $c(M) = \operatorname{glb}\{c(\eta)\colon \eta \in Z^1(M)\}$. If M is a symplectic manifold with symplectic form ω, then there is a C^1-neighborhood of id_M in $\operatorname{Diff}(M, \omega)$, so that if f is in this neighborhood, then $V(f)$ is a closed one-form. Furthermore, the number of fixed points of f is equal to $c(V(f))$. Now assume M is simply connected, so that every closed one-form is exact. Then $c(M) \geqslant 2$ since every smooth function on a compact manifold has at least two critical points. Therefore, in this C^1-neighborhood of id_M every f has at least two fixed points.

2. The main theorem. When the manifold M is not compact there are functions with no critical points, and hence there are closed one-forms with no zeros. Therefore, $c(M) = 0$. In this section we extend the fixed point theorem of Weinstein to open symplectic manifolds. Note that while M may be a symplectic manifold, its completion $\tilde{M}$ may carry no symplectic structure at all. In particular, for the open

$2n$-cell $B^{2n} = \{x \in \mathbf{R}^{2n}\colon \|x\| < 1\}$ the completion is homeomorphic to S^{2n}, which has no symplectic structure for $n > 1$. The open manifold B^{2n} has the standard symplectic structure induced from $\mathbf{R}^{2n}$.

THEOREM 2.1. *If (M, w) is a symplectic manifold with $e(M) < c(\tilde{M})$, then there exists a C^1-fine neighborhood A of* id_M *in* $\mathrm{Diff}(M, \omega)$ *such that every $f \in A$ has at least $c(\tilde{M}) - e(M)$ fixed points.*

PROOF. Assume M is embedded in $\tilde{M}$ as an open submanifold. Let $\phi\colon \tilde{M} \to \mathbf{R}$ be a nonnegative function vanishing only on the ends of M, $\phi(x) = 0$ if and only if $x \in \tilde{M} - M$. Let $B \subset Z^1(M)$ be the set of one-forms defined by ϕ,

$$B = \{\eta \in Z^1(M)\colon \|\eta(x)\| < \phi(x), \|D\eta(x)\| < \phi(x)\}$$

where the norms arise from a riemannian metric on $\tilde{M}$. So B is an open subset *and* every $\eta \in B$ extends to a form $\tilde{\eta}$ on $\tilde{M}$ such that $\tilde{\eta}(x) = 0$ for $x \in \tilde{M} - M$. By taking an intersection, if necessary, we may assume that B satisfies the conclusions of Theorem 1.1. Since $c(\tilde{M}) - e(M) > 0$ and $c(\tilde{\eta}) \geqslant c(\tilde{M})$, it follows that $c(\tilde{\eta}) - e(M) > 0$, so that $\tilde{\eta}$ has more zeros than there are points in $\tilde{M} - M$. Therefore $\eta(x) = 0$ for some $x \in M$. Now we use Theorem 1.1 to get a C^1-fine neighborhood A in $\mathrm{Diff}(M, \omega)$ containing the identity and a homomorphism $V\colon A \to B$. For $f \in A$, the one-form $V(f)$ is in B and so f has a fixed point x in M. $\square$

We now restrict our attention to manifolds M diffeomorphic to $\mathbf{R}^{2n}$. Let ω be any symplectic structure on M. Clearly, $e(M) = 1$ and by picking a point $N \in S^{2n}$, we can embed M onto $S^{2n} - \{N\}$, so that $\tilde{M} \approx S^{2n}$. With this construction and the fact that $c(S^{2n}) = 2$, we have

COROLLARY 2.2. *Let (M, ω) be a symplectic manifold where M is diffeomorphic to $\mathbf{R}^{2n}$. Then there is a neighborhood in the C^1-fine topology of* $\mathrm{Diff}(M, \omega)$ *which contains* id_M, *such that every mapping in this neighborhood has a fixed point.*

One should be aware that there are symplectic diffeomorphisms of $\mathbf{R}^{2n}$ with symplectic structure $\sum dx_i \wedge dy_i$ that have no fixed points, in particular the translations, but there are C^1-fine neighborhoods of the identity containing no translations. Let $\phi\colon \mathbf{R}^{2n} \to \mathbf{R}^+$ be a function vanishing at infinity and use ϕ to define an open neighborhood consisting of the diffeomorphisms f such that $\|f(x) - x\| < \phi(x)$ and $\|Df(x) - I\| < \phi(x)$ for all $x \in \mathbf{R}^{2n}$.

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