A DUALITY PRINCIPLE

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ABSTRACT. With the aid of the Baire category theory we prove an extension of Erdős' well-known duality principle concerning sets of Lebesgue measure zero and sets of first category.

Assuming the continuum hypothesis, Sierpiński [9] proved the existence of a bijective function \( f: \mathbb{R} \to \mathbb{R} \), such that \( A \) is of first category iff \( f(A) \) has Lebesgue measure zero. By a modification of Sierpiński's proof, Erdős [2] showed that the function \( f \) in Sierpiński's result can be chosen such that \( f = f^{-1} \). The importance of these results is that they allow one to state a well-known duality principle (see [6]). Using Morgan’s abstract Baire category theory (cf. [3–5]), Cholewa [1] generalized Sierpiński’s theorem, but an analogous generalization of Erdős’ theorem is not known.

The aim of this paper is to give a short proof of a new extension of Erdős' result.

In this note we assume, that \( X \) is a topological group, which is a complete, separable, metric space without isolated points; moreover we suppose that the reader is familiar with Morgan’s theory.

**Theorem 1.** Let \( \mathfrak{c} = \omega_1 \). If \( \mathfrak{C} \) and \( \mathcal{D} \) are nonequivalent \( \mathfrak{S} \)-families \([4]\) and \( \mathfrak{E} \)-families on \( X \), satisfying c.c.c. (countable chain condition), then there is a bijective function \( f: X \to X \) such that \( f = f^{-1} \) and such that \( A \in \mathfrak{C} \) iff \( f(A) \in \mathcal{D} \).

**Proof.** The properties of \( X \) imply that |\( X | = \mathfrak{c} \), since otherwise

\[
X = \bigcup \{ \{x\} : x \in X \}
\]

would be of first category. We note that \( \mathfrak{C}_1 \) is a \( \sigma \)-ideal such that \( X = \bigcup \mathfrak{C}_1 \), since \( \{x\} \in \mathfrak{C}_1 \) for all \( x \in X \) [4, Theorem 6]. Now let

\[
\mathfrak{g} := \{ A \in \mathfrak{C}_\delta \cap \mathfrak{C}_1 : |A| > \aleph_0 \},
\]

where \( \mathfrak{C} \) denotes the family of all sets being complements of \( \mathfrak{C} \)-sets. By Theorem 3 in [5] and by Corollary 10 in [4] each \( \mathfrak{C}_1 \)-set is contained in some \( \mathfrak{g} \)-set. Since \( \mathfrak{C} \) consists of perfect sets, we get \( |\mathfrak{C}| \leq \mathfrak{c} \) and thus \( |\mathfrak{g}| \leq \mathfrak{c} \). Moreover, the complement of a \( \mathfrak{C}_1 \)-set lies in \( \mathfrak{B}(\mathfrak{C}) \cap \mathfrak{C}_\Pi \) and by Corollary 15 in [4], it contains a nonempty \( \mathfrak{C} \)-singular perfect set, that is, it contains a \( \mathfrak{C}_1 \)-set of power \( \mathfrak{c} \) (Lemma 3 in [4] shows that, without loss of generality, we can assume, that \( X \in \mathfrak{C} \subset \mathfrak{C}_\Pi \)).

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Thus $\mathcal{C}$ fulfills the properties (a)–(d) of Theorem 19.5 in [6]. It is clear, that the same fact is true for $\mathcal{D}$. Since $\mathcal{C}$ and $\mathcal{D}$ are nonequivalent, it follows from Theorem 2 in [8], that $X$ is the disjoint union of a $\mathcal{C}$-set and a $\mathcal{D}$-set. Thus, by Theorem 19.6 in [6], the proof is finished.

The classical result of Erdős can be obtained from Theorem 1 by setting $\mathcal{C} := \{\{x \in \mathbb{R} : |x - y| \leq 1/n\} : y \in \mathbb{R}, \ n \in \mathbb{N}\}$ and $\mathcal{D} := \{A \subset \mathbb{R} : A \text{ is closed, } \forall x \in A \ \forall U \in \tau x : \mu(U \cap A) > 0\} \ (\tau x \text{ consists of all open subsets of } \mathbb{R} \text{ containing } x \text{ and } \mu \text{ denotes the Borel measure in } \mathbb{R})$. It follows, that $\mathcal{C}$ and $\mathcal{D}$ are nonequivalent $\mathfrak{B}$- and $\mathfrak{S}$-families (see [4 and 8]). Since the members of $\mathcal{C}$ and $\mathcal{D}$ have positive measure, $\mathcal{C}$ and $\mathcal{D}$ also satisfy c.c.c. [7, p. 123].

REFERENCES

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