QUOTIENTS OF BANACH SPACES OF COTYPE $q$

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Abstract. Let $Z$ be a Banach space and let $X \subset Z$ be a $B$-convex subspace (equivalently, assume that $X$ does not contain $l_1^m$'s uniformly). Then every Bernoulli series $\sum_{n=1}^{\infty} \varepsilon_n z_n$ which converges almost surely in the quotient $Z/X$ can be lifted to a Bernoulli series a.s. convergent in $Z$. As a corollary, if $Z$ is of cotype $q$, then $Z/X$ is also of cotype $q$. This extends a result of [4] concerning the particular case $Z = L_1$.

In this note, we give some further applications of the main results of [5]. It is well known that, in general, an unconditionally convergent series in a quotient Banach space $Z/X$ cannot be lifted up to an unconditionally convergent series in $Z$. However, we prove in this note that if $X$ is $B$-convex, then a similar lifting property holds for "almost" unconditionally convergent series. (Of course, this becomes trivial if the subspace $X$ is complemented in $Z$.)

We will need some specific notations: let $D = \{-1, 1\}^N, \varepsilon_n: D \to \{-1, +1\}$ the $n$th coordinate, and let $\mu$ denote the uniform probability on $D$ (i.e. $\mu$ is the normalized Haar measure on the compact group $D$). For any finite set of integers $A \subset N$, we denote by $W_A$ the Walsh function $W_A = \prod_{n \in A} \varepsilon_n$. These functions form an orthonormal basis of characters of the space $L_2(D, \mu)$. We will denote by $R_k$ the orthogonal projection from $L_2(D, \mu)$ onto the closed linear span of the functions $\{W_A | |A| = k\}$; moreover, we will denote by $\mathcal{P}$ the linear span of all the functions of the form $W_A$ (i.e. $\mathcal{P}$ is the space of all “trigonometric polynomials” on the group $D$). It is well known that, for each $\varepsilon$ in $[0, 1]$, the operator $T(\varepsilon) = \sum_{k \geq 0} \varepsilon^k R_k$ (defined a priori only on $\mathcal{P}$) extends to a contraction on $L_p(D, \mu)$ for each $p$ such that $1 < p < \infty$.

Now let $Z$ be an arbitrary Banach space. We denote by $I_Z$ the identity operator on $Z$. Obviously the operator $T(\varepsilon) \otimes I_Z$ (defined a priori only on $\mathcal{P} \otimes Z$) extends to a linear contraction—which we still denote $T(\varepsilon) \otimes I_Z$—on the space $L_p(D, \mu; Z)$ for $1 < p < \infty$. For simplicity, we will write in the sequel $L_p(Z)$ instead of $L_p(D, \mu; Z)$. We will denote also $B(L_p(Z))$, the Banach space of all bounded operators on $L_p(Z)$.

Now let $X$ be a $B$-convex Banach space (equivalently a space which does not contain $l_1^m$'s uniformly, see [3] for details). It was proved in [5] that there exists a constant $C > 1$ (depending only on $X$) such that the operator $R_k \otimes I_X$ (defined a
priori only on $\mathcal{Q} \otimes X$) defines a bounded operator on $L_2(X)$ and verifies
\[ \forall k > 0, \quad \| R_k \otimes I_X \|_{\mathcal{B}(L_2(X))} \leq C^k. \]

We can now show the main result of this note.

**Theorem.** Let $Z$ be an arbitrary Banach space and let $X \subset Z$ be a $B$-convex closed subspace. Denote by $\pi: Z \to Z/X$ the canonical surjection. Then for any sequence $(y_n)$ in $Z/X$ such that the series $\sum_{n=1}^{\infty} \varepsilon_n y_n$ converges in $L_2(Z/X)$, we can find a sequence $(z_n)$ in $Z$, such that $\pi(z_n) = y_n$ for each $n$, and the series $\sum_{n=1}^{\infty} \varepsilon_n z_n$ converges in $L_2(Z)$ and verifies
\[ \| \sum_{n=1}^{\infty} \varepsilon_n z_n \|_{L_2(Z)} \leq K \left( \sum_{n=1}^{\infty} \varepsilon_n y_n \right)_{L_2(Z/X)}, \]
for some constant $K$ depending only on $X$.

**Proof.** Obviously, it is enough to prove the theorem assuming that $(y_n)$ is a finite sequence $(y_1, \ldots, y_N)$ (so that $y_n = 0$ for all $n > N$). Denote by $P_N$ the set of all finite subsets of $\{1, 2, \ldots, N\}$.

Assume that $\| \sum_{n=1}^{N} \varepsilon_n y_n \|_{L_2(Z/X)} < 1$.

By an obvious pointwise lifting, we can find a function $\Phi: D \to Z$, depending only on the $N$ first coordinates, and such that $\| \Phi \|_{L_2(Z)} < 1$ and
\[ \pi(\Phi) = \sum_{n=1}^{N} \varepsilon_n y_n. \]

We may as well assume that $\Phi$ is an odd function on $D$, i.e. that $\Phi(\omega) = -\Phi(-\omega)$ $\forall \omega \in D$ (otherwise, we replace $\Phi(\omega)$ by $\frac{1}{2}(\Phi(\omega) - \Phi(-\omega))$).

A priori, $\Phi$ admits a development as follows:
\[ \Phi = \sum_{A \in P_N} W_A z_A \quad \text{with} \quad z_A \in Z. \]

($z_A$ is defined as $\int \Phi W_A \ d\mu$.)

The equality (3) implies that $\pi(z_{(n)}) = y_n$ $\forall n = 1, 2, \ldots, N$, and
\[ \pi(z_A) = 0 \quad \text{if} \quad |A| \neq 1. \]

Therefore, $z_A$ belongs to $X$ whenever $|A| \neq 1$; moreover, since $\Phi$ is odd, we have $z_A = 0$ whenever $|A|$ is even.

We claim that the sequence $(z_{(1)}, \ldots, z_{(N)})$ verifies the desired property:
Let us write for simplicity,
\[ \Phi_k = R_k \otimes I_Z(\Phi), \]
so that $\Phi = \sum_{k \geq 0; \text{odd}} \Phi_k$, and $\Phi_k = 0$ for all even $k$.

Since $T(\varepsilon) \otimes I_Z$ is a contraction on $L_2(Z)$, and since $T(\varepsilon) \otimes I_Z(\Phi) = \sum_{k \geq 0} \varepsilon^k \Phi_k$, we have
\[ \left\| \sum_{k \geq 0} \varepsilon^k \Phi_k \right\|_{L_2(Z)} < 1. \]
It follows that
\[ \epsilon \| \Phi_1 \|_{L_2(Z)} < 1 + \sum_{k > 3} \epsilon^k \Phi_k \|_{L_2(Z)}. \]

But by (4) we know that \( \Phi_k \) actually belongs to \( L_2(X) \) for each \( k > 3 \), and we may as well assume that \( X \) verifies (1), so that we have, for each \( \epsilon < 1/C \),
\[ \| \sum_{k > 3} \epsilon^k \Phi_k \|_{L_2(X)} < \sum_{k > 3} \epsilon^k C^k \| \Phi - \Phi_1 \|_{L_2(X)} \]
\[ < (\epsilon C)^3 (1 - \epsilon C)^{-1} (1 + \| \Phi_1 \|_{L_2(Z)}). \]

Combining (5) and (6), we find
\[ \| \Phi_1 \|_{L_2(Z)} < (\epsilon C)^3 (1 - \epsilon C)^{-1} (1 + \| \Phi_1 \|_{L_2(Z)}). \]

Now, if we choose \( \epsilon \) such that \( 2C(\epsilon C)^2 = 1/2 \), i.e. \( \epsilon = (4C^3)^{-1/2} \), we have \( (1 - \epsilon C)^{-1} < 2 \) and (7) yields
\[ \| \Phi_1 \|_{L_2(Z)} < K, \]
with \( K = 2(1 + \frac{1}{2}(4C^3)^{-1/2})(4C^3)^{1/2} \).

By homogeneity, this concludes the proof of (2) in the finite case, and hence completes the proof of the theorem.

A Banach space \( Z \) is called of cotype \( q \) if there is a constant \( \lambda \) such that, for any finite sequence \( (z_n) \) in \( Z \), we have
\[ \left( \sum \| z_n \|^{q} \right)^{1/q} < \lambda \| \sum \epsilon_n z_n \|_{L_2(Z)}. \]
(See [3] for more details on this notion.)

It is well known that \( L_1 \)-spaces are of cotype 2. It is also well known that, in general, a quotient of a cotype \( q \) space need not be of cotype \( q \). However, it was proved in [4] that if \( R \) is a \( B \)-convex subspace of \( L_1 \), then \( L_1/R \) is of cotype 2. (Without any restriction on \( R \), this is certainly false since for example \( c_0 \) is isometric to a quotient of \( l_1 \).) The next corollary generalizes this last result.

**Corollary 1.** In the situation of the theorem, if \( Z \) is of cotype \( q \), then \( Z/X \) is also of cotype \( q \).

**Corollary 2.** Assume that \( X \subset Z \) does not contain \( l_1^n \)'s uniformly. If \( Z \) does not contain \( l_1^n \)'s uniformly, the same is true for \( Z/X \).

**Proof.** This follows from the theorem and Theorem 1.1 in [3].

It is natural to ask whether there exists a “geometric” proof of the last result. Also, there might be an infinite dimensional analogue concerning e.g. spaces which do not contain \( l_1 \). Moreover, we do not know, in the situation of the theorem, whether or not we can lift unconditionally convergent series in \( Z/X \) into unconditionally convergent series in \( Z \).

The reader will have noticed that, when \( X \) is a “concrete” Banach space, for instance when \( X \) is a Hilbert space, or when \( X \) embeds in \( L_p \) for some \( 1 < p < \infty \),
then the results of [5] are not needed in the proof of the above theorem. Therefore, we have also obtained a more direct proof of the result of [4], that $L_1/R$ is of cotype 2 whenever $R$ is a reflexive subspace of $L_1$. (Recall that, by [6], $R \subset L_1$ is reflexive iff $R$ is $B$-convex and in that case $R$ embeds in $L_p$ for some $p > 1$.)

**REMARKS.** We mention here some easy generalizations of the previous results:

(i) By a result of Kahane (cf. [2], p. 17), for any $0 < p < \infty$ and any Banach space $Z$, a Bernoulli series $\sum_{n=1}^{\infty} e_{zn}$ is convergent in $L_p(Z)$ iff it converges a.s. in $Z$. Therefore, we may also state the theorem as it is in the abstract.

(ii) In the same situation as in the theorem, we can prove using the same basic idea: For each $k > 0$ there is a constant $K(k)$ such that, for every integer $N$, every set $\{y_A | A \in P_N, |A| = k\}$ in $Z/X$ can be lifted to a subset $\{z_A | A \in P_N, |A| = k\}$ of $Z$, which verifies

$$\left\| \sum_{A \in P_N} W_A z_A \right\|_{L_2(Z)} < K(k) \left\| \sum_{A \in P_N} W_A y_A \right\|_{L_2(Z/X)}.$$

Finally it is easy to generalize the proof of the above theorem as follows:

**COROLLARY 3.** In the situation of the theorem, let $(Y_n)$ be a sequence of independent random variables with values in $Z/X$. Fix $p$ such that $1 < p < \infty$. Assume that $\sum_{n=1}^{\infty} Y_n$ converges a.s. (resp. in $L_p(Z/X)$); then there exist a sequence $(Z_n)$ of independent $Z$-valued random variables, such that $Z_n$ is $Y_n$-measurable, $\sum_{n=1}^{\infty} Z_n$ converges a.s. (resp. in $L_p(Z)$), and $\pi(Z_n) = Y_n$.

In the preceding statement, we implicitly assume that all the random variables considered have a separable range.

**PROOF.** The part concerning the convergence in $L_p(Z)$ can be proved exactly as in the theorem but using the proof of Corollary 3.4 instead of Theorem 2.1 in [5]. More precisely, let $\mathcal{E}_n$ be the $\sigma$-algebra generated by $\{Y_m | m \neq n\}$, and let $V_k$ be the projection defined on $L_p(d\mathbb{P})$ by

$$V_k = \sum_{A \subset N} \prod_{j \in A} (I - E_{\mathcal{E}_j}) \prod_{j \notin A} E_{\mathcal{E}_j}.$$

The proof of Corollary 3.4 in [5] shows that if $X$ is $B$-convex, then there exists a constant $C$ such that

$$\forall k > 1 \quad \left\| V_k \otimes I_X \right\|_{B(L_p(X))} < C^k.$$

The first part of the proof can then be completed by reasoning as we did to prove the theorem.

Now, if $\sum_{n=1}^{\infty} Y_n$ converges a.s. in $Z/X$, we define $Y'_n = Y_{n,1}(\|Y_n\| < 1)$. By Corollary 3.3 in [1], the series $\sum_{n=1}^{\infty} Y'_n$ converges in $L_p(Z/X)$ for each $p < \infty$. Therefore by the first part of the proof we can find, for each $n$, a $Z$-valued variable $Z'_n$ which is $Y'_n$-measurable and such that $\sum_{n=1}^{\infty} Z'_n$ converges (say) in $L_2(Z)$ and $\pi(Z'_n) = Y'_n$. Since the variables $(Z'_n)$ are independent, $\sum_{n=1}^{\infty} Z'_n$ converges also a.s. By lifting trivially $Y_n - Y'_n$, we can find a $Y_n$-measurable $Z$-valued variable $Z''_n$ supported by the set $\{\|Y_n\| > 1\}$ and such that $\pi(Z''_n) = Y_n - Y'_n$. 

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Clearly, if we define $Z_n = Z'_n + Z''_n$, the series $\sum_{n=1}^{\infty} Z_n$ converges a.s. in $Z$ and $\pi(Z_n) = Y_n$ for each $n$. Q.E.D.

Remark. Let $(g_n)_{n \geq 1}$ be a sequence of independent, equidistributed, standard Gaussian variables. Using Remark 2.2 in [5], it is easy to prove the above theorem with the sequence $(g_n)$ instead of $(e_n)$. By known results on Gaussian measures, this yields immediately: In the situation of the theorem, any Gaussian Radon measure on $Z/X$ is the image (by the canonical surjection) of a Gaussian Radon measure on $Z$.

REFERENCES


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