Oscillation Criteria for the Sublinear Schrödinger Equation

HIROSHI ONOSE

Abstract. In this paper, we propose an oscillation theorem to the sublinear Emden-Fowler equation \( \Delta u + g(x)u^\gamma = 0 \) for \( n = 2 \). By using this, we will answer to the open problem of Noussair and Swanson in the case of bounded solutions.

1. Introduction. Recently, Noussair and Swanson [6, 7, 8] have developed some oscillation criteria for the Schrödinger equation. We also refer to Kreith and Travis [4] and others [1, 3, 5] for related results. Especially, Noussair and Swanson [7] get a necessary and sufficient condition for nonoscillation of a sublinear Emden-Fowler equation in the case of \( n > 3 \). But, the important case of \( n = 2 \) has not been settled up to date. In this paper, we propose an oscillation theorem for the case of \( n = 2 \).

The partial differential equations to be examined are of the form

\[
\Delta u + \sum_{i=1}^{n} g_i(x) |u|^{\gamma_i} \text{sgn } u = 0,
\]

(1)

where \( x \in \Omega, 0 < \gamma_1 < 1 \) and \( \gamma_i > 0 \) for \( i = 2, 3, \ldots, n \), and \( g_i(x) \in C(\Omega) \) for \( i = 1, 2, \ldots, n \). Equation (1) is said to be oscillatory in \( \Omega \) if every solution \( u(x) \in C^2(\Omega) \) of (1) is nontrivial in any neighbourhood of infinity has arbitrarily large zeros: that is, the set \( \{ x \in \Omega: u(x) = 0 \} \) is unbounded. Equation (1) is called nonoscillatory in an unbounded domain \( \Omega \) whenever (1) has a one-signed solution in \( \Omega \cap G_r \) for some positive number \( r \). Let \( |x| \) denote the Euclidean norm of a point \( x = (x_1, x_2) \) in a real 2-dimensional Euclidean space \( \mathbb{R}^2 \). Define

\[
S_r = \{ x \in \mathbb{R}^2: |x| = r \}, \quad G_r = \{ x \in \mathbb{R}^2: |x| > r \}
\]

and

\[
G(r, s) = \{ x \in \mathbb{R}^2: r < |x| < s \}, \quad 0 < r < s.
\]

We define a function \( f(r) \) on \( (0, \infty) \) by the equation

\[
\bar{u}(r) = f(r) = \frac{1}{|S_r|} \int_{S_r} u(x) \, dS, \quad |S_r| = 2\pi r,
\]

which is a spherical mean of the function \( u(x) \), and we define a function \( \check{u}(r) = \sup_{|x| = r} u(x) \).
2. The results.

**THEOREM 1.** Consider the equation

\[ \Delta u + \sum_{i=1}^{n} g_i(x) \left| u \right|^\gamma_i \text{sgn} u = 0 \quad \text{in } G_a, \]

where \(0 < \gamma_i < 1, \gamma_i > 0\) and \(g_i(x) \geq 0\) in \(G_a\) for \(i = 2, 3, \ldots, n\). If

\[ \int_{r_0}^{\infty} r (\log r)^{\gamma_1} g_1(r) \, dr = +\infty \quad \text{for some } r_0 > 0, \]

then every bounded solution of (2) is oscillatory.

**Proof.** Suppose to the contrary that there exists a solution \(u(x)\) of (2) which has no zero in \(G_b\) for some \(b > a > 0\). We may assume that \(u(x) > 0\) in \(G_b\), since a parallel argument holds if \(u(x) < 0\) in \(G_b\). We write the \(\gamma_i\) in (2) simply by \(\gamma\). We may take a constant \(k\) such that

\[ 0 < \gamma < k < 1 \quad \text{and} \quad 1 - k < \gamma. \]

By the hypothesis that \(g_i(x) > 0\) in \(G_b\), we have \(\Delta u \leq 0\) in \(\Omega_b\). As \(u(x)\) is bounded, it follows that

\[ \bar{u}(t) < c \quad \text{for } t > T_1 > b \quad \text{and a constant } c > 0. \]

Put \(v(x) = (\log |x|)^\gamma, w = u^{1-k}\). By using the following Green’s formula

\[ \int_{G(T_1, t)} (v \Delta w - w \Delta v) \, dx = \int_{\partial G(T_1, t)} (v \frac{\partial w}{\partial n} - w \frac{\partial v}{\partial n}) \, dS, \]

we obtain

\[ \int_{T_1}^{t} \left( r (\log r)^{\gamma} \bar{w} + \gamma (1 - \gamma) \frac{\bar{w}}{r (\log r)^{2-\gamma}} \right) \, dr \]

\[ = t (\log t)^{\gamma} \bar{w}(t) - \gamma (\log t)^{\gamma - 1} \bar{w}(t) + c_1 \]

\[ = t (\log t)^{2\gamma} \frac{d}{dt} \left[ \frac{\bar{w}(t)}{(\log t)^{\gamma}} \right] + c_1 \quad \text{where } c_1 \text{ is a constant.} \]

It follows from (4) and (5) that

\[ \bar{w} = \frac{u^{1-k}}{(\bar{u})^{1-k}} \leq c^{1-k} \equiv c_2 \quad \text{for } t > T_1. \]

By using (7), we have

\[ 0 < \int_{T_1}^{\infty} \frac{\bar{w}}{r (\log r)^{2-\gamma}} \, dr \leq c_2 \int_{T_1}^{\infty} \frac{dr}{r (\log r)^{2-\gamma}} \]

\[ = \frac{c_2}{1 - \gamma} (\log T_1)^{\gamma - 1} < +\infty. \]

We compute

\[ \Delta w = (1 - k) \frac{\Delta u}{u^k} - k (1 - k) \frac{\nabla u^2}{u^{1+k}} \leq (1 - k) \frac{u^\gamma}{u^k} \leq -c_3 g_1(x), \]
where $c_3$ is a positive constant and $\gamma - k$ is negative, and we used the boundedness of $u(x)$. From (9) and (3), we deduce that the left-hand integral of (6) tends to $-\infty$ as $t \to \infty$. By using (8) and (9), we obtain

$$-1 \geq t \left( \frac{\log t}{2} \right)^{2\gamma} \frac{d}{dt} \left[ \frac{w(t)}{\left( \log t \right)^{\gamma}} \right] \quad \text{for } t \geq T_2 \geq T_1.$$  

By integrating (10) over $[T, t]$, for $T > \max \{ T_2, \epsilon \}$, we have

$$\log \frac{w(t)}{w(T)} \leq \int_T^t \frac{dr}{r \left( \log r \right)^{2\gamma}}.$$  

Let $t \to \infty$ in (11). If $1 - 2\gamma > 0$, then we have an immediate contradiction to the boundedness of $\tilde{w}(t)$. If $1 - 2\gamma < 0$, then (11) implies $\tilde{w}(T) \geq c_4 \left( \log T \right)^{1-\gamma}$ for sufficiently large $T$ and a constant $c_4$. Since $1 - \gamma > 0$, we again have a contradiction. Q.E.D.

A special case of Theorem 1 is the following

**Corollary.** Consider the sublinear equation

$$\Delta u + g(x) |u|^{\gamma} \text{ sgn } u = 0 \quad \text{in } G_a \text{ where } 0 < \gamma < 1 \text{ and } g(x) > 0 \text{ in } G_a.$$  

If

$$\int_{r_0}^{\infty} r \left( \log r \right)^{\gamma} \tilde{g}(r) \, dr = +\infty \quad \text{for some } r_0 > 0,$$

then every bounded solution of (12) is oscillatory.

**Theorem 2** [7, Theorem 5.2]. Suppose

1. $g(x) > 0$ for all $x \in \Omega$, $g \in C^\alpha(\bar{M})$ for every bounded subdomain $M \subset \Omega$, and $\hat{g} \in C^\alpha[a, b]$ for some $a > 0$, $0 < a < 1$ and for all $b > a$.

The sublinear Emden-Fowler equation (12) is nonoscillatory in an exterior domain in $\mathbb{R}^2$ if

$$\int_{r_0}^{\infty} r \left( \log r \right)^{\gamma} \hat{g}(r) \, dr < \infty \quad \text{for some } r_0 > 0.$$

Our extra hypothesis is

$$\liminf_{r \to \infty} \frac{\hat{g}(r)}{\hat{g}(r)} > 0.$$  

**Theorem 3.** If $g(x)$ satisfies (15) and the hypothesis of Theorem 2, then (13) is a necessary and sufficient condition for all bounded solutions of the sublinear Emden-Fowler equation (12) to be oscillatory in an exterior domain $\Omega \subset \mathbb{R}^2$.
PROOF. The sufficiency is contained in the Corollary. If (13) fails, then by (15) there exist positive numbers $e$ and $r_0$ such that
\[ +\infty > \int_{r_0}^{\infty} r (\log r)^{7/6} \hat{g}(r) \, dr > e \int_{r_0}^{\infty} r (\log r)^{7/6} \hat{g}(r) \, dr, \]
whence all solutions of (12) are nonoscillatory by Theorem 2. Q.E.D.

EXAMPLE. Consider the equation
\[ \Delta u + \frac{1}{r^2 (\log r)^{7/6}} u^{1/3} = 0. \]

Since
\[ \int_{r_0}^{\infty} r (\log r)^{1/3} \frac{1}{r^2 (\log r)^{7/6}} \, dr = \int_{r_0}^{\infty} \frac{1}{r (\log r)^{3/6}} \, dr = \int_{r_0}^{\infty} X^{-5/6} \, dX = +\infty, \]
every bounded solution of (16) is oscillatory by Theorem 1.

REMARK. We compute
\[ \int_{r_0}^{\infty} \frac{1}{r (\log r)^{3/6}} \, dr = \int_{r_0}^{\infty} \frac{1}{r (\log r)^{7/6}} \, dr = \int_{r_0}^{\infty} X^{-7/6} \, dX < +\infty. \]

Equations (17) and (18) show that equation (16) is not contained in the results of Kitamura and Kusano [2] but is covered by Theorem 1.

ACKNOWLEDGEMENT. The author wishes to thank the referee for several helpful comments.

REFERENCES


DEPARTMENT OF MATHEMATICS, IBARAKI UNIVERSITY, MITO 310, JAPAN