A SPACE OF POINTWISE COUNTABLE TYPE AND PERFECT MAPS

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Abstract. There exists a Lindelöf space, of pointwise countable type, which does not admit a perfect map onto any space in which every point is $G_δ$.

Introduction. Recall from [A] that a space is of pointwise countable type if each point is contained in a compact set of countable character. In [O, Question 7.5], Olson asked if there is a paracompact Hausdorff space, of pointwise countable type, which does not admit a perfect map onto a first countable space. In this note, we answer this question affirmatively by exhibiting an example of a regular Lindelöf space, of pointwise countable type, which does not admit a perfect map onto any space in which every point is $G_δ$. The example is obtained by adding the closed unit interval to the space $M$ constructed by Dowker in [D].

The example. Let $I$ be the closed unit interval, and $ω_1$ the first uncountable ordinal. For ordinals $α, β$ with $α ≤ β ≤ ω_1$, $[α, β]$ denotes the space $\{γ | α < γ ≤ β\}$ of ordinals with the order topology. Let $Q$ be the set of all rational numbers in $I$, and let $\{Q_α | α < ω_1\}$ be a disjoint collection of countable dense subsets in $I$ consisting of irrational numbers. Consider the product space $[0, ω_1] × I$ and its subspace

$$X = ([0, ω_1] × I) \setminus \bigcup_{a < ω_1} ([0, a] × Q_a).$$

Claim 1. $X$ is a Lindelöf space of pointwise countable type.

Proof. Let $\mathcal{U}$ be an open cover of $X$. By using compactness of $\{ω_1\} × I$, we can find $α_0 < ω_1$ such that $([α_0, ω_1] × I) \cap X$ is covered by finitely many members of $\mathcal{U}$. Since $([0, α_0] × I) \cap X$ satisfies the second axiom of countability, it follows that $\mathcal{U}$ has a countable subcover. For each $(α, p) ∈ X$, $([0, ω_1] × \{p\}) \cap X$ is a compact set of countable character and contains $(α, p)$. □

Let $Y$ be a space in which every point is $G_δ$, and let $f: X → Y$ be a continuous map such that $f^{-1}(y)$ is compact for each $y ∈ Y$. It suffices to show that $f$ is not a closed map. For each $q ∈ Q$, since $f^{-1}f((ω_1, q))$ is a $G_δ$-set, there is $α_q < ω_1$ such that $[α_q, ω_1] × \{q\} ⊆ f^{-1}f((ω_1, q))$. Let $β = \sup\{α_q | q ∈ Q\}$; then $β < ω_1$. Let us set $J = \{ω_1\} × I$.

Claim 2. For each $y ∈ Y$, $f^{-1}(y) \cap J$ is nowhere dense in $J$. 

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Proof. Suppose that $f^{-1}(y) \cap J$ contains an open interval $U$ in $J$. Then there exist $p \in Q_{\beta}$ with $(\omega_1, p) \in U$ and a sequence $(q_n)_{n \in \mathbb{N}}$ in $Q$, converging to $p$, such that $(\omega_1, q_n) \in U$ for each $n \in \mathbb{N}$. Let $E = \{(\beta, q_n) \mid n \in \mathbb{N}\}$. Then $E \subset f^{-1}(y)$ and $E$ is discrete closed in $X$ since $(\beta, p) \notin X$. This contradicts the fact that $f^{-1}(y)$ is compact. □

Claim 3. $f$ is not a closed map.

Proof. Pick $r \in Q_{\beta}$, and let $y = f((\omega_1, r))$. By Claim 2, we can find a sequence $(s_n)_{n \in \mathbb{N}}$ in $Q$ such that $|r - s_n| < 1/n$ and

$$(\omega_1, s_n) \notin f^{-1}(y).$$

Let $y_n = f((\omega_1, s_n))$, for each $n \in \mathbb{N}$, and $F = \{(\beta, s_n) \mid n \in \mathbb{N}\}$. Then, $f$ being continuous, $(y_n)$ converges to $y$. Since $(\beta, r) \notin X$, $F$ is closed in $X$. But $f(F)$ is not closed in $Y$, because $f(F) = \{y_n \mid n \in \mathbb{N}\}$. Hence the proof is complete. □

Remarks. Olson's question was repeated by Burke in [B], and appears also in [R]. It is not possible to strengthen our example by making $X$ locally compact. In fact, $X$ would then be mapped perfectly onto a metrizable space (cf. [F, Theorem 3]).

References


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