ON SEMIPRIME RINGS OF BOUNDED INDEX

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ABSTRACT. A ring $R$ is of bounded index (of nilpotency) if there is an integer $n \geq 1$ such that $x^n = 0$ whenever $x \in R$ is nilpotent. The least such positive integer is the index of $R$. We show that a semiprime ring $R$ has index $\leq n$ if and only if $R$ is a subdirect product of prime rings of index $\leq n$.

A ring $R$ is of bounded index if there is an integer $n \geq 1$ such that $x^n = 0$ whenever $x$ is a nilpotent element of $R$. The least such positive integer is called the index of $R$, and we denote it by $i(R)$. V. A. Andrunakievič and Ju. M. Rjabuhin [1], and, independently, P. N. Stewart [6] have shown that a ring of index 1 (i.e. a ring with no nonzero nilpotent elements) is a subdirect product of integral domains. In this article we will show that a semiprime ring of index $n$ is a subdirect product of prime rings each of index at most $n$.

For a subset $X$ of a ring $R$ we let $r(X)$ denote the right annihilator of $X$, while $l(X)$ denotes the left annihilator of $X$. If $T$ is an ideal of a semiprime ring $R$, then it is easily seen that $l(T) = r(T)$; hence we write $\text{ann}(T)$ for this two-sided ideal of $R$.

We begin by adding one more equivalence to the following result of J. Hannah [4, Proposition 2].

\textbf{Lemma 1.} Let $R$ be a semiprime ring, $n$ a positive integer. Then the following statements are equivalent.

(a) $i(R) \leq n$.

(b) $\forall X_1, \ldots, X_n \subseteq R$ such that $X_iX_j = 0$ whenever $i \geq j$, then $X_1X_2 \cdots X_n = 0$.

(c) If $X \subseteq R$, then $r(X^n) = r(X^{n+1})$.

(d) For each $x \in R$, $r(x^n) = r(x^{n+1})$.

\textbf{Lemma 2.} Let $R$ be a semiprime ring and $n$ a positive integer. Then $i(R) \leq n$ if and only if

(*) for each $x \in R$ and each ideal $T$ of $R$, $r(Tx^n) = r(Tx^{n+1})$.

\textbf{Proof.} If (*) holds, then letting $T = R$ we get $r(Rx^n) = r(Rx^{n+1})$. However $R$ is semiprime so $r(Ra) = r(a)$ for all $a \in R$. Hence (d) of Lemma 1 is satisfied so that $i(R) \leq n$. Thus assume $i(R) \leq n$. We shall use Lemma 1(b) to establish (*). Proceeding as in Hannah’s proof, we let $X_i = r(Tx^i)Tx^i$ for $i = 1, 2, \ldots, n$. Then for $i \geq j$ we have $X_iX_j = 0$ so we must have

$$0 = X_1X_2 \cdots X_n = T \cdot X_1X_2 \cdots X_n$$

$$= [T \cdot r(Tx)][Tx \cdot r(Tx^2)][Tx^2 \cdot r(Tx^3)] \cdots [Tx^{n-1} \cdot r(Tx^n)]Tx^n.$$
Now for each $i \geq 0$ (with $T^{x_0} = T$) we have $T^{x_i} \cdot r(T^{x_{i+1}}) = T^{x_i} \cdot x^{n-i} \cdot r(T^{x_{i+1}})$ and $x^{n-i} \cdot r(T^{x_{i+1}}) \subseteq r(T^{x_{i+1}})$. Hence $T^{x_i} \cdot r(T^{x_{i+1}}) \subseteq T^{x_i} \cdot r(T^{x_{i+1}})$ for $i \geq 0$. But then $[T^{x_i} \cdot r(T^{x_{i+1}})]^{n+1} = 0$ so $T^{x_i} \cdot r(T^{x_{i+1}}) = 0$ since $R$ is semiprime. Thus $r(T^{x_{i+1}}) \subseteq r(T^{x_i})$ and equality follows.

**Lemma 3.** Assume $R$ is a semiprime ring of index $n$ and $T$ is an ideal of $R$. If $K = \text{ann}(T)$ then $i(R/K) \leq n$.

**Proof.** Since $K = \text{ann}(T)$, $R/K$ is a semiprime ring. Let $\tilde{R} = R/K$ and let $\tilde{a} = a + K \in \tilde{R}$. If $\tilde{w} \in r(\tilde{a}^{n+1})$, then $a^{n+1}w \in K$ so $T^{a^{n+1}}w = 0$. Since $r(T^{a^{n+1}}) = r(T^{a^n})$, we have $a^{n}w \in K$ and so $\tilde{w} \in r(\tilde{a}^{n})$. By Lemma 1(c), $i(R/K) \leq n$.

**Theorem 1.** Let $R$ be a semiprime ring of index $n$. Then $R$ is a subdirect product of prime rings of index $\leq n$.

**Proof.** We must show that for each $a \in R$, $a \neq 0$, there is a prime ideal $P$ with $a \notin P$ and $i(R/P) \leq n$. Since $R$ is semiprime and $a \neq 0$, there is an $m$-system containing $a$ [5, Chapter 4]. Specifically let $a_1 = a$; then $a_1R_1a_1 \neq 0$ implies $a_1b_1a_1 = a_2 \neq 0$ for some $b_1 \in R$. Inductively we obtain sequences $\{b_i\}$, $\{a_i\}$ with $a_{i+1} = a_i b_i a_i \neq 0$ and $M = \{a_i\}$ is the desired $m$-system. Using Zorn's Lemma, the set $\{A \mid A$ an ideal of $R$, $M \cap A = \emptyset$ and $i(R/A) \leq n\}$ has a maximal element $P$ which we claim is a prime ideal of $R$. To show that $P$ is a prime ideal, we first note that $P$ is a semiprime ideal of $R$. For if $P \subseteq W$ and $W^2 \subseteq P$, then $i(R/W) \leq n$ since $i(R/P) \leq n$. Now if $P \subseteq W$, then $M \cap W \neq \emptyset$ so $a_i \in W$ for some $i \geq 1$. But then $a_iR_1a_1 \subseteq W^2 \subseteq P$, so $a_{i+1} \in P$, a contradiction. Thus $P$ is indeed a semiprime ideal of $R$. By passing to $R/P$ we assume that $P = 0$. If $R$ is semiprime but not prime, we have nonzero ideals $A, B$ with $A = \text{ann}(B)$ and $B = \text{ann}(A)$. Then by Lemma 3, $i(R/A) \leq n$ and $i(R/B) \leq n$. Hence $M \cap A \neq \emptyset$ and $M \cap B \neq \emptyset$. But if $a_i \in M \cap A$, $a_j \in M \cap B$ with $j \geq i$, then $a_j \in A \cap B = 0$, a contradiction. Thus $P$ is a prime ideal, as desired.

As a consequence we have the index 1 result.

**Theorem 2 (Andrunakievic and Rjabuhin; Stewart).** A ring $R$ has no zero nilpotent elements if and only if $R$ is a subdirect product of integral domains.

In addition we deduce

**Theorem 3.** Let $R$ be a ring of index $n$, and let $N$ denote the prime radical of $R$. Then $N = \cap\{P \mid P$ is a prime ideal, $i(R/P) \leq n\}$.

**Proof.** Since $N$ is a nil ideal, $i(R/N) \leq n$ and $R/N$ is semiprime so the statement follows.

Hannah also establishes in [4, Proposition 5] that a prime ring has index $\leq n$ if and only if each chain of left (right) annihilators has at most $n$ proper inclusions. Thus a prime ring of bounded index has ACC on left and right annihilators. As Hannah points out, however, semiprime rings of index $n$ need not have the ACC on left or right annihilators. (An infinite direct product of fields provides an appropriate example.) There is, however, a class of semiprime rings for which bounded index yields the ACC on left and right annihilators. A semiprime ring with unit is (right) strongly semiprime if each faithful ideal contains a finitely generated left ideal whose right annihilator is zero. A strongly prime ring is a prime ring
which is strongly semiprime. For properties of these rings see [2, 3]. In particular D. Handelman shows [3, Theorem 1] that a ring $R$ is strongly semiprime if and only if $R$ is a finite subdirect product of strongly prime rings. Lemma 3 is instrumental in showing

**Theorem 4.** A ring $R$ is strongly semiprime of index $\leq n$ if and only if $R$ is a finite subdirect product of strongly prime rings of index $\leq n$.

**Proof.** According to [3, Theorem 1] the prime rings are obtained by taking $R/P$, where $P$ is maximal among the annihilators of two-sided ideals of $R$. If $R$ has index $n$, then, by Lemma 3, $i(R/P) \leq n$. The converse is evident.

Using [4, Proposition 5] yields

**Theorem 5.** If $R$ is a strongly semiprime ring of index $n$ then $R$ has ACC on left and right annihilators.

**References**


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