RATIONAL NONLINEAR CHARACTERS
OF METABELIAN GROUPS

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ABSTRACT. Let G be a finite metabelian group with all nonlinear irreducible characters rational. Then the exponent of the commutator group G' is a prime or divides 16, 24, or 40. If G' is also cyclic, then its order is a prime or divides 12.

1. Introduction. Let G be a finite group having all characters (over the complex field) rational. Markel [6] proved that if G is supersolvable then G has order $|G| = 2^t 3^s$. In [4] Gow proved that if G is solvable then $|G| = 2^t 3^s 5^h$, and in [7] Vishnevetskij proved similar results if G is finite with some added conditions.

In this paper we assume that G is a finite metabelian group having all nonlinear (absolutely) irreducible characters rational. After proving a few preliminary results we prove, in §3, that the exponent of G' is either a prime or divides 16, 24, or 40. Assuming that G' is cyclic we prove, in §4, that $|G'|$ is a prime or it divides 12. In particular if all the characters of G are rational, then the results of this paper imply that $|G| = 2^t 3^s$.

2. Preliminary results. Let G be a finite group. By a character of G we shall always mean a character over the complex field, and by an irreducible character we shall always mean an absolutely irreducible character. A character χ of G is rational (real) if $\chi(g)$ is rational (real) for all $g \in G$. If K is a normal subgroup of G and $a \in G$, then $\langle a \rangle$ and $\langle aK \rangle$ denote the cyclic subgroups of G and G/K respectively, and $|a|_0$ and $|aK|_0$ denote the orders of $\langle a \rangle$ and $\langle aK \rangle$ respectively. Also $\exp(G)$ denotes the exponent of G. If A is a group that acts on a group B then $\mathbf{C}_A(B) = \{x \in A | b^x = b \text{ for all } b \in B\}$.

LEMMA 1. *Let G be a finite group, K a normal subgroup of G, χ a linear character of K such that χ^G is irreducible, $L = \ker \chi$, and $N(L)$ the normalizer of L in G. Then χ^G is real if and only if some element of $N(L)$ inverts every element of K/L and χ^G is rational if and only if $N(L)/K \simeq \mathrm{Aut}(K/L)$, the automorphism group of K/L.*

PROOF. Let χ_1 be the $\mathbf{Q}$-conjugate (complex conjugate) of χ. Then $\ker \chi_1 = L$, χ_1^G is irreducible and is $\mathbf{Q}$-conjugate (complex conjugate) to χ^G. From [3, (45.5)], $\chi_1^G = \chi^G$ if and only if χ_1 and χ are G-conjugate. The results follow.

A somewhat deeper result concerning the realness of χ^G is given in [2, §2].

Now assume G is a finite metabelian group, A a normal subgroup of G such that A and G/A are abelian. Let χ be a linear character of A with $\ker \chi = L$, and $K(L)$ a subgroup of $N(L)$ containing A such that $K(L)/L$ is a maximal abelian subgroup

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of $N(L)/L$. The χ has $|K(L)/A|$ extensions χ' to $K(L)$ and each χ'^G is irreducible and is a constituent of χ^G. The characters χ'^G, χ running over all linear characters of A, form all the irreducible characters of G. See [1]. Every extension of χ to $K(L)$ is given by $\mu\chi'$, where χ' is a fixed extension of χ to $K(L)$ and μ is a linear character of $K(L)$ with $\ker\mu \supseteq A$. Note that if $L_1 = \ker\chi'$ then $L_1 \cap A = L$ and $N(L_1) \subseteq N(L)$. Lemma 1 and the above results give

COROLLARY. *Let G be a finite group, A a normal subgroup of G, A and G/A abelian, and let χ be a linear character of A. If some irreducible constituent of χ^G is real, then some element of $N(L)$ inverts all elements of A/L. If some irreducible constituent of χ^G is rational, then $N(L)$ induces the full automorphism group on A/L.*

We omit the proof of the next result.

LEMMA 2. *Let U be a cyclic group of order m and let $A = \mathrm{Aut}(U)$. For subgroups V of U let $B(V) = \mathbf{C}_A(U/V)$. Assume for every V if $|U/V| = p$, p an odd prime, then $B(V)$ has exponent dividing 2, and if $|U/V| = 2$ then $B(V)$ has exponent dividing 4. Then either m is a prime or m divides 16, 24, or 40.*

3. The main result. We need the following

LEMMA 3. *Let A be a normal subgroup of finite group G, G/A abelian, and let χ be a linear character of A such that all the irreducible constituents of χ^G are real. Let $I = I_G(\chi)$, the inertia group of χ in G and $L = \ker\chi$. Then*
(a) *$|I'L/L| \leq 2$.*
(b) *If $I' \subseteq L$ then $\exp(I/A) \leq 2$.*
(c) *If $\chi^2 = 1_A$ then $\exp(I/A)$ divides 4.*

PROOF. Since $G'' \subseteq L$, we may assume G to be metabelian. The elements of I commute (modulo L) with the elements of A, and for any $y \in N(L)$ and $x \in I$, $y^{-1}xy \equiv xa_x \,(\mathrm{mod}\,L)$, $a_x \in A$. Thus for any $x, w \in I$ and $y \in N(L)$, $y^{-1}[x,w]y \equiv [x,w]\,(\mathrm{mod}\,L)$, where $[x,w] = x^{-1}w^{-1}xy$. Thus $I'L/L$ is in the center of $N(L)/L$. Also $I'L/L$ is cyclic since A/L is cyclic. The realness of the constituents of χ^G implies that some element of $N(L)$ inverts $I'L/L$. Thus $|I'L/L| \leq 2$.

Now let χ' be an extension of χ to $K(L) \subseteq I$, $\theta = \chi'^I$, an irreducible constituent of χ^I, and μ be a linear character of I with $\ker\mu \supseteq A$. If μ' is an extension of μ to G then $\mu\theta = (\mu_{K(L)}\chi')^I$ and $\mu'\chi'^G = (\mu\theta)^G = (\mu_{K(L)}\chi')^G$, that is, $\mu\theta$ and $\mu'\theta^G$ are constituents of χ^I and χ^G respectively. Now $\theta^G = \bar\theta^G$ and $(\mu\theta)^G = (\bar\mu\bar\theta)^G$, bars denoting the complex conjugates. Thus

$$(\bar\mu\bar\theta)^G = \bar\mu'\bar\theta^G = \bar\mu'\theta^G = (\bar\mu'\theta)^G \text{ or } (\mu\theta)^G = (\bar\mu\theta)^G.$$

From Isaacs [5, 6.11], $\mu\theta = \bar\mu\theta$. Assuming $I' \subseteq L$, we have $I = K(L)$, θ linear, and hence $\mu = \bar\mu$ for every linear character μ of I, $\ker\mu \supseteq A$. Thus $\exp(I/A) \leq 2$.

Now assume $\chi^2 = 1_A$ and $\chi \neq 1_A$ (for if $\chi = 1_A$ then $I' \subseteq L$). Thus $|A/L| = 2$ and $N(L) = I \supseteq K(L) \supseteq A$. If $y \in I$ and $k \in K(L)$, then $y^{-1}ky \equiv ka \,(\mathrm{mod}\,L)$ for some $a \in A$, and hence $y^2 \in K(L)$. Any extension of χ to $K(L)$ is given by $\mu\chi'$ where μ is any linear character of $K(L)$, $\ker\mu \supseteq A$, and χ' is a fixed extension of χ to $K(L)$. Using the above, the realness of $(\mu\chi')^G$ implies $\mu(k^2)\chi'(k^2) = \pm1$. Setting $\mu = 1_{K(L)}$ we get $\chi'(k^2) = \pm1$. Thus $\mu(k^2) = \pm1$ for every μ or $\exp(K(L)/A)$ divides 4. Now assume for some $k \in K(L)$, k of order 4 modulo A, and some $y \in I$,

$y \notin K(L)$ and $y^2 \equiv k \pmod{A}$. Then we may pick another $K(L)$, say $K_1(L)$, such that $y \in K_1(L)$ or $\exp(K_1(L)/A) = 8$, a contradiction. Thus the above results imply $\exp(I/A)$ divides 4 completing the proof.

COROLLARY. *Let G be a finite metabelian group having all nonlinear irreducible characters real and for any subgroup H of G', $G'/H \neq 1$, let $t(H) = \exp(\mathbf{C}_G(G'/H)/G')$. Then $t(H)$ divides 4 if $|G'/H|$ is a power of 2 and $t(H) \leq 2$ otherwise.*

PROOF. If $G'/H \neq 1$, then G' has a linear character χ, $H \subseteq \ker\chi \subset G'$ and $|G'/\ker\chi|$ a prime. The result follows from the above lemma.

THEOREM 1. *Let G be a finite metabelian group with all irreducible nonlinear characters rational and let $m = \exp(G')$. Then either m is a prime or m divides 16, 24, or 40.*

PROOF. Let χ be a linear character of G', $L = \ker\chi$, $|G'/L| = m$. Then $N(L)$ induces the full automorphism group on $U = G'/L$. Let $p|m$, p a prime, and let $\lambda = \chi^{m/p}$. Then $S = \ker\lambda \supseteq L$ and letting $V = S/L$ we have $|U/V| = p$. Let $I = I_G(\lambda)$ then $I \cap N(L) = \mathbf{C}_{N(L)}(U/V)$.

Assume $p > 2$. From Lemma 3(a), $|I'S/S| \leq 2$, and hence $I' \subseteq S$. Thus Lemma 3(b) implies $\exp(I/G') \leq 2$. If $p = 2$, then $\lambda^2 = 1_{G'}$, and from Lemma 3(c), $\exp(I/G')$ divides 4. Now $N(L)/\mathbf{C}_{N(L)}(U) \simeq \mathrm{Aut}(U)$ and thus $I \cap N(L)/\mathbf{C}_{N(L)}(U) \simeq B(V)$ in the notation of Lemma 2. Since $\mathbf{C}_{N(L)}(U) \supseteq G'$, we have $\exp(B(V)) \leq 2$ if $|U/V| = p$, an odd prime, and $\exp(B(V))$ divides 4 if $|U/V| = 2$. Now applying Lemma 2, we have m a prime or m dividing 16, 24, or 40. The proof is complete.

If the linear characters of a finite group G are rational, then G/G' is an elementary abelian 2-group. The above results give

COROLLARY. *The order of a finite metabelian group with all characters rational is of the form $2^t 3^s$.*

We give an example of a metabelian group G with $\exp(G') = 24$ and all nonlinear irreducible characters rational.

Let $\langle a, b \rangle$ and $\langle x, y, z \rangle$ be abelian groups with $a^8 = b^8 = 1$ and $x^2 = y^2 = z^2 = 1$, and let $G_1 = \langle a, b \rangle \circ \langle x, y, z \rangle$, the semidirect product, with the action of $\langle x, y, z \rangle$ on $\langle a, b \rangle$ defined by

$$a^x = a^{-1}, \quad b^x = b^{-1}, \quad a^y = a^5, \quad b^y = b^5, \quad a^z = b, \quad b^z = a.$$

Then $G_1' = \langle a^{-1}b, a^2 \rangle$ and $\exp(G_1') = 8$. It is easy to show that all the irreducible characters of G_1 are rational. Since all the irreducible characters of the dihedral group D_3 of order 6 are rational, the group $G = G_1 \times D_3$ has all irreducible characters rational and $\exp(G') = 24$.

The group $G_2 = \langle a, b, x \rangle$ with defining relations $a^2 = b^4 = x^2 = 1$, $ab = ba$, $ax = xa$, and $x^{-1}bx = ab$ has all nonlinear irreducible characters rational and two linear characters complex. It is not known if metabelian groups G with $\exp(G') = 16$ or 40 exist that have all nonlinear irreducible characters rational.

4. Cyclic commutator group. We prove

THEOREM 2. *Let G be a finite metabelian group with all nonlinear irreducible characters rational and assume the commutator group G' is cyclic. Then the order of G' is a prime or divides 12.*

PROOF. From Theorem 1, if $|G'|$ is odd then $|G'|$ is a prime. Thus we assume $|G'|$ is even and prove our result by taking the case when $|G'|$ is a power of 2, and then the case when $|G'|$ is even but not a power of 2.

Assume that $G' = \langle a \rangle$ is cyclic of order 2^s. Theorem 1 implies that $s \leq 4$. We shall prove that $s \leq 2$ by studying $\exp(K(1)/G')$ which from Lemma 3 divides 4. Here $K(1) = K(\langle 1 \rangle)$ is as in the notations of §2.

Assuming for some $K(1)$ the exponent of $K(1)/G'$ is 4, we get the following two cases.

(i) There is b in some $K(1)$, $|bG'|_0 = 4$ and $b^4 = 1$.

(ii) There is b in some $K(1)$, $|bG'|_0 = 4$ and $b^4 = a^2$.

Note that if $b^4 = a$, then some nonlinear irreducible character of G is not real, and if $b^4 = a^{2^t}$, $t > 1$, then setting $b_0 = b^{-1}a^{2^{t-2}}$, we have $b_0^4 = 1$, which is case (i).

Assuming for every $K(1)$, $K(1)/G'$ is of exponent ≤ 2, we get two additional cases.

(iii) $K(1)/G'$ is of exponent 2 for every $K(1)$, and there is c in some $K(1)$, $|cG'|_0 = 2$ and $c^2 = a$.

(iv) For every $K(1)$, $K(1)/G'$ is of exponent dividing 2 and $K(1)$ splits over G'.

Note that if $c^2 = a^{2^t}$, $t > 0$, then setting $c_0 = c^{-1}a^{2^{t-1}}$ gives c_0^2, which is case (iv). Also (iv) includes the case where $K(1) = G'$.

We shall prove our result by studying separately each of the above four cases.

Case (i). Let $L = \langle a^2 \rangle$ and pick $K(L) \supseteq \langle a, c \rangle$. Let χ be a linear character of $K(L)$ such that $\chi(b) = \sqrt{-1}$, $S \cap G' = L$, where $S = \ker \chi$. Since χ^G is rational, we have $z \in N(S)$ such that $\chi(z^{-1}bz) = \chi(b^{-1}) = -\sqrt{-1}$. But $b^{-1}z^{-1}bz$ is in $G' = \langle a \rangle$ implying $z^{-1}bz = ba^f$, f an odd integer. Thus $(ba^f)^4 = 1$ or $a^{4f} = 1$ and $s \leq 2$.

Case (ii). Consider $K(1) \supseteq \langle a, b \rangle$ and let χ be a linear character of $K(1)$ with $S = \ker \chi$ and $S \cap G' = 1$. Then $K(1)/S = \langle bS \rangle$ and $\chi(b)^4 = \chi(a^2)$. Assume $\chi(b)^2 = \chi(a)$. (The other choice for χ is given by $\chi(b)^2 = -\chi(a)$.) Since χ^G is rational we have $N(S)/K(1) = \langle yK(1), zK(1) \rangle$ such that $y^{-1}by \equiv b^5 \pmod{S}$ and $z^{-1}bz \equiv b^{-1} \pmod{S}$. Since $|bS|_0 = 2^{s+1}$ we have $|yK(1)|_0 = 2^{s-1}$ or $|yG'|_0 = 2^{s-1}$, 2^s, or 2^{s+1}. But since $b^{-1}y^{-1}by \equiv b^4 \equiv a^2 \pmod{S}$ and $S \cap G' = 1$, we have $y^{-1}by = ba^2$. Let $L_1 = \langle a^2 \rangle$; then $y^{-1}by \equiv b \pmod{L_1}$, and hence we may pick $K(L_1) \supseteq \langle a, b, y \rangle$, not necessarily containing $K(1)$. Thus $|yG'|_0$ divides 4 or $s \leq 3$. Assume $s = 3$, and let y be as above. Then $y^4 = 1$ or $y^4 = a^2$ or $y^4 = a^4$. If $y^4 = a^2$ then by letting $y_0 = yb$ we get $y_0^4 = 1$. If $y^4 = a^4$ then by letting $y_1 = ya$ we get $y_1^4 = 1$. That is, we may assume $y^4 = 1$. Let χ_1 be a linear character of $K(L_1)$, $S_1 = \ker \chi_1$, $S_1 \cap G' = \langle a^2 \rangle$, and $\chi_1(y) = \sqrt{-1}$. Then there is $w \in N(S_1)$ such that $w^{-1}yw = ya^f$, f an odd integer. But then $(ya^f)^4 = a^{4f} \neq 1$, a contradiction since $y^4 = 1$. Thus $s \leq 2$.

Case (iii). With slight variations, the proof of this case is similar to case (ii) above in which c replaces b.

Case (iv). Assume every $K(1)$ can be written as $\langle a, c_1, \ldots, c_t \rangle$ with $a^{2^s} = c_i^2 = 1$, $i = 1, \ldots, t$. If $s = 4$ then by a method similar to case (ii), where $s = 3$, in which a replaces b, we get a likewise contradiction. Thus assume $s = 3$.

Since $G' = \langle a \rangle$, there are g and h in G such that $g^{-1}h^{-1}gh = a^e$, e odd. (Without loss of generality we may take $e = 1$.) Thus $h^{-1}gh = ga$. We also have

$g^{-1}ag = a^\tau$ and $h^{-1}ah = a^\sigma$, τ and σ odd integers. First assume $\tau = 1$. Then we may pick $K(1)$ such that $g \in K(1)$ and hence either $g^2 = 1$ or $(ga^\lambda)^2 = 1$, for some λ. Exchanging $g_0 = ga^\lambda$ for g gives $h^{-1}g_0h = g_0a^f$, f odd. But $(g_0a^f)^2 \neq 1$, a contradiction. Thus $\tau \neq 1$. Similarly $\sigma \neq 1$. Thus τ and σ may take the values 3, 5, or 7 (mod 8). If $\tau = \sigma$ then letting $g_0 = gh$ we have $g_0a = ag_0$ and get a contradiction as above. Thus $\tau \neq \sigma$, and hence one of τ, σ, or $\tau\sigma$ takes the value 5 (mod 8). Without loss of generality we assume $\tau = 5$ and $\sigma = -1 \equiv 7$ (mod 8). That is, we have $g^{-1}ag = a^5$ and $h^{-1}ah = a^{-1}$. Since $h^{-1}gh = ga$ we have $(ga)^2 = g^2a^6$ and $(ga)^4 = g^4a^4$.

Let $L = \langle a^4 \rangle$. Then $g^{-1}ag \equiv a \pmod{L}$, and thus we may pick $K(L)$ such that $g \in K(L)$. This implies that either $|gG'|_0 = 2$ or $|gG'|_0 = 4$. Assume $|gG'|_0 = 2$; then we have $g^2 = 1$, $g^2 = a^2$, or $g^2 = a^4$. Then the orders of $h^{-1}gh = ga$ and of g are not equal, a contradiction. Thus $|gG'|_0 = 4$ which implies $g^4 = 1$, $g^4 = a^4$, or $g^4 = a^2$. Again in the first two cases, the orders of $h^{-1}gh = ga$ and of g are not equal, a contradiction. This leaves us with the only case $g^4 = a^2$.

Let χ be a linear character of $K(L) \supseteq \langle a, b \rangle$, $S = \ker \chi$, $S \cap G' = L$, and assume $\chi(g)^2 = \chi(a)$. Then $K(L)/S = \langle gS \rangle$ and thus $N(S)/K(L) = \langle yK(L), zK(L) \rangle$ with $y^{-1}gy \equiv g^5 \pmod{S}$ and $z^{-1}gz \equiv g^{-1} \pmod{S'}$. Thus $y^{-1}gy \equiv ga^2 \pmod{L}$ and $z^{-1}gz \equiv ga^{-1} \pmod{L}$. Since $y^{-1}ay \equiv a \pmod{L}$, it follows that $y^{-1}ay = a$ or $y^{-1}ay = a^5$. If $y^{-1}ay = a^5$, then letting $y_0 = yg$, we get $y_0^{-1}ay_0 = a$. Thus, we may assume $y^{-1}ay = a$. Hence we may pick some $K(1)$ such that $y \in K(1)$. But the assumptions on $K(1)$ imply that either $y^2 = 1$ or $(ya^\lambda)^2 = 1$ for some λ. Without loss of generality, we may assume $y^2 = 1$. Since $y^{-1}gy \equiv ga^2 \pmod{L}$, we have $y^{-1}gy = ga^2$ or ga^{-2}. These give $g^{-1}yg = ya^{-2}$ or ya^2. But then $(g^{-1}yg)^2 \neq 1$, a contradiction. This completes the proof of case (iv) and the proof of the first part of the theorem.

Now assume $|G'|$ is even but not a power of 2. From Theorem 1 only 3 or 5 may divide $|G'|$. We make a slightly more general assumption, that is, G' has a cyclic 5-Sylow subgroup and a not necessarily cyclic 2-Sylow subgroup, and then arrive at contradictions.

Let $G' = A \times B$, A an abelian 2-group, $B = \langle b \rangle$, $|B| = 5$. Let L be a (fixed) subgroup of A, $|A/L| = 2$, $A/L = \langle aL \rangle$ and $B = \langle b \rangle$. Then $G'/L = \langle abL \rangle$ is of order 10. Assuming all the nonlinear irreducible characters to be rational we have $\exp(K(L)/G') \leq 2$.

We have the following two cases:

(v) For every $K(L)$, $K(L)/L$ splits over G'/L, i.e., $K(L)/L = G'/L \times E/L$, E/L an elementary abelian 2-group.

(vi) There exists some c in some $K(L)$ such that $c^2 \equiv a \pmod{L}$.

Case (v). Assume (v) and fix $K(L)$. Let χ be a linear character of $K(L)$, $\ker \chi = L_1$, $L_1 \cap G' = L$. Then $N(L_1) \subseteq N(L)$ and $N(L_1)/K(L) \simeq \text{Aut}\langle K(L)/L_1 \rangle$, $N(L_1)/K(L) = \langle yK(L) \rangle$ is of order 4, and $y^{-1}ay \equiv a \pmod{L}$, $y^{-1}by \equiv b^2 \pmod{L}$ and $y^{-1}zy \equiv z \pmod{L}$ for all $z \in E$. Now let $\overline{L} = \langle L, B \rangle = L \times B$. Since y commutes (mod $\overline{L}$) with all elements of $K(L)$, we choose $K(\overline{L}) \supseteq \langle K(L), y \rangle = N(L_1)$. It could easily be shown that $K(\overline{L}) = N(L_1)$ and $N(\overline{L}) = N(L)$. If $y^4 \equiv a \pmod{\overline{L}}$, then some nonlinear character of $N(\overline{L})$ (and hence of G) is not real. Since $y^2 \notin G'$ we have $y^4 \equiv 1 \pmod{\overline{L}}$, and hence we may assume $y^4 \equiv 1 \pmod{L}$. Thus $K(\overline{L})/\overline{L}$ splits over $G'/\overline{L}$, i.e., $K(\overline{L})/\overline{L} = G'/\overline{L} \times \langle E, y, b \rangle/\overline{L}$.

Let θ be a linear character of $K(\overline{L})$ with $\overline{L}_1 = \ker\theta$, $\overline{L}_1 \cap G' = \overline{L}$, and $\theta(y) = \sqrt{-1}$. Let $x \in N(\overline{L}_1)$; then $x \in N(\overline{L}) = N(L)$ and $x^{-1}zx \equiv z \pmod{\overline{L}}$ for all $z \in E$. Thus $x^{-1}zx \equiv z \pmod{L}$ for all $z \in E$, and hence $x \in N(L_1) = K(\overline{L})$. Thus $N(\overline{L}_1) = K(\overline{L})$, and since θ is not real, θ^G is not real, a contradiction.

Case (vi). In this case we may assume $K(L)/L = \langle cL \rangle \times B \times E/L$, E/L an elementary abelian 2-group, and $G'/L = \langle c^2, B \rangle / L$. If χ is a linear character of $K(L)$, $\ker\chi = L_1$, $L_1 \cap G' = L$, and $\chi(c) = \sqrt{-1}$, then $K(L)/L_1 = \langle cbL_1 \rangle$, of order 20, and $N(L_1)/K(L) = \langle xK(L), yK(L) \rangle$ with

$$x^{-1}cx \equiv c^{-1}, \quad y^{-1}by \equiv b^2, \quad x^{-1}bx \equiv b,$$

$$y^{-1}cy \equiv c, \quad x^{-1}zx \equiv y^{-1}zy \equiv z,$$

with all congruences modulo L and $z \in E$. Let $\overline{L} = \langle L, B \rangle$ and pick $K(\overline{L}) \supseteq \langle K(L), y \rangle$. Then $K(\overline{L}) = \langle K(L), y \rangle$. Let θ be a linear character of $K(\overline{L})$, $\overline{L}_1 = \ker\theta$ and $\overline{L}_1 \cap G' = \overline{L}$. If $y^4 \equiv a \pmod{\overline{L}}$, then θ^G is not real. Assume $y^4 \equiv 1 \pmod{\overline{L}}$ and hence assume $y^4 \equiv 1 \pmod{L}$. Then $K(\overline{L}) \subseteq N(\overline{L}_1) \subseteq N(L_1) = \langle K(\overline{L}), x \rangle$. If $x^{-1}yx \equiv y \pmod{L}$, then let $\theta(c) = \theta(y) = \sqrt{-1}$. Then $x^{-1}(c^{-1}y)x \equiv cy \pmod{L}$, and since $\theta(c^{-1}y) = 1$ and $\theta(cy) = -1$, we have $x \notin N(\overline{L}_1)$. Thus $N(\overline{L}_1) = k(\overline{L})$ and θ^G is not real. If $x^{-1}yx \equiv ya \pmod{L}$ then let $\theta(y) = 1$. Since $\theta(ya) = -1$, again $N(\overline{L}_1) = K(\overline{L})$ and θ^G is not real. These contradictions complete the proof of this part and proof of the theorem.

The above proof also gives

COROLLARY. *Assume all the nonlinear irreducible characters of the metabelian group G are rational and G' is of even order. If the p-Sylow subgroup of G' is cyclic where p is an odd prime, then $p = 3$.*

We conclude with

COROLLARY. *Assume every nonlinear irreducible character of G is rational and G' is cyclic. If $|G'| = p$, p an odd prime, then $|G| = 2^t p(p-1)$; if $|G'| = 2$ or 4 then $|G| = 2^t$, and if $|G'| = 6$ or 12 then $|G| = 2^t \cdot 3$.*

BIBLIOGRAPHY

1. B. G. Basmaji, *Monomial representations and metabelian groups*, Nagoya Math. J. **35** (1969), 99–107. MR **39** #5709.
2. ――――, *Representations of metabelian groups realizable in the real field*, Trans. Amer. Math. Soc. **169** (1972), 389–399. MR **46** #9153.
3. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. XI, Interscience, New York, 1962. MR **26** #2519.
4. R. Gow, *Groups whose characters are rational-valued*, J. Algebra **40** (1976), 286–299. MR **53** #13372.
5. I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976. MR **57** #417.
6. F. M. Markel, *Groups with many conjugate elements*, J. Algebra **26** (1973), 69–74. MR **48** #8624.
7. A. L. Vishnevetskij, *Finite groups with rational characters*, Dokl. Akad. Nauk Ukrain. SSR, Ser. A (1977), no. 10, 876–878, 958. (Russian) MR **57** #9822.

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