ON FUNCTIONS IN THE BALL ALGEBRA

P. WOJTASZCZYK

ABSTRACT. We show that there exists a function in a ball algebra such that almost every slice function has a series of Taylor coefficients divergent with every power \( p < 2 \).

In §7.2 of [3] W. Rudin gives some examples of boundary behavior of holomorphic functions in complex balls of dimension 2 and 3. It was observed in [4, Remark 1.10], that using Theorem 1.2 of [4] such examples can be constructed in arbitrary dimension. In the present note we further pursue this idea. It is well known that in one variable there exists a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) analytic for \( |z| < 1 \) such that \( \sum |a_n|^p = \infty \) for all \( p < 2 \). Our theorem generalizes this fact to functions of several variables.

In this note, \( B \) will always denote the unit ball in the complex \( d \)-dimensional space \( \mathbb{C}^d \), \( S \) will stand for the unit sphere and \( \sigma \) for the rotation invariant probability measure on \( S \). \( A(B) \) will denote the ball algebra of all functions analytic in \( B \) and continuous in \( \overline{B} \). For \( 0 < p \leq \infty \), \( ||f||_p \) denotes \( (\int_S |f(z)|^p d\sigma(z))^{1/p} \). If \( f \) is a holomorphic function in \( B \) then it has a unique homogeneous expansion as \( f = \sum_{n=0}^{\infty} f_n \) where \( f_n \) is an analytic polynomial homogeneous of degree \( n \).

It was shown in [4, Theorem 1.2], that there exist polynomials \( (p_n) \) homogeneous of degree \( n \) on \( B \) such that

\[
(*) \quad ||p_n|| = 1 \quad \text{and} \quad ||p_n||_\infty \leq \chi.
\]

(\( \chi \) depends only on the dimension of the ball, it can be taken \( \chi = 2^d/\sqrt{n} \)). Those polynomials will be crucial in our further considerations. In particular, we will use the following inequality (cf. [4, Proposition 1.6])

\[
(**) \quad \left\| \sum_{k=0}^{\infty} a_k p_{2n} \right\|_p \sim \left( \sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2}, \quad 0 < p < \infty.
\]

We will also use the fact that Cauchy Integral \( C[\mu] \) maps continuously measures on \( S \) into all \( H_p(B) \), \( p < 1 \) (cf. [3, 6.2]).

The book [3] is an excellent source of information about the function theory in \( B \).

**PROPOSITION.** The operator \( T : A(B) \to \ell_2 \) defined by \( T(f) = ((f, p_{2n}))_{n=0}^{\infty} \) is a surjective map.

**PROOF.** This map is clearly continuous. By duality it is enough to show that \( T^* : \ell_2 \to A(B)^* \) is an isomorphic embedding i.e. \( ||T^*(a_n)||_{A(B)^*} \geq C(\sum |a_n|^2)^{1/2} \).
One easily checks that \( T^*(\alpha_n)(f) = \langle f, \sum_{n=0}^{\infty} \alpha_n p_{2^n} \rangle \). Let \( \mu \) be a measure on \( S \) such that, for \( f \in A(B) \), \( T^*(\alpha_n)(f) = \int f \, d\mu \) and \( ||\mu|| = ||T^*(\alpha_n)|| \). Then obviously \( C[\mu] \), the Cauchy integral of \( \mu \), equals \( \sum_{n=0}^{\infty} \alpha_n p_{2^n} \), so (by properties of \( C[\cdot] \) and \((**))

\[
||T^*(\alpha_n)|| = ||\mu|| = ||\mu|| \geq C||C[\mu]||_{1/2} = C \left\| \sum_{n=0}^{\infty} \alpha_n p_{2^n} \right\|_{1/2} \geq C \left( \sum_{n=0}^{\infty} |\alpha_n|^2 \right)^{1/2}.
\]

**Corollary.** There exists a constant \( C \) such that for every sequence \( (\alpha_n)_{n=0}^{\infty} \) with \( \alpha_n = 0 \) for \( n \leq K \) and \( n \geq N \) (for some integers \( K \) and \( N \)) there exists \( f \in A(B) \) such that

\[
||f||_{\infty} \leq C \left( \sum_{n=0}^{\infty} |\alpha_n|^2 \right)^{1/2}, \quad Tf = (\alpha_n),
\]

and the homogeneous expansion of \( f \) has a form

\[
\sum_{s=2^K+1}^{2^{N-1}} f_s.
\]

**Proof.** Let \( \beta \) be a piecewise linear function given by

\[
\beta(t) = \begin{cases} 
0 & \text{if } t < 2^K \text{ or } t \geq 2^N, \\
1 & \text{if } 2^K+1 \leq t \leq 2^{N-1}, \\
\text{linear} & \text{otherwise}.
\end{cases}
\]

It follows from the classical de la Vallée-Poussin theorem that the map \( f = \sum_{n=0}^{\infty} f_n \mapsto \sum_{n=0}^{\infty} \beta(n) \cdot f_n \) maps \( A(B) \) continuously into \( A(B) \), and has norm bounded by an absolute constant. This observation and the proposition proves the corollary.

**Theorem.** There exists a function \( f \in A(B) \) with the homogeneous expansion \( f = \sum_{n=0}^{\infty} f_n \) such that for every \( p < 2 \) and \( \sigma \)-almost all \( \xi \in S \), \( \sum_{n=0}^{\infty} |f_n(\xi)|^p = \infty \) i.e., almost every slice function of \( f \) has a series of Taylor coefficients divergent with every power \( < 2 \).

**Proof.** Let us denote \( 2^{2^r+k} \) by \( \gamma(r,k) \). Repeated use of the corollary gives a sequence of functions \( g^r \in A(B) \) such that

1. The homogeneous expansion of \( g^r \) has form

\[
g^r = \sum_{\gamma(r,0)+1}^{\gamma(r+1,0)-1} g^r_k;
\]

2. \( ||g^r||_{\infty} \leq C; \)

3. \((g^r, p_{\gamma(r,k)}) = 2^{-r/2}, k = 1, 2, \ldots, 2^r - 1.\)

The properties (1)–(3) imply that there exist two constants \( \alpha, \beta \) such that for every \( r \) and \( p \)

\[
\sum_{\sigma} |g^r_k(\xi)|^p \geq C 2^{r(1-p/2)} \text{ on a set } A_r \subset S \text{ with } \sigma(A_r) \geq \beta.
\]

**Proof of (4).** By (3) and \((\ast)\) we infer
Since \( C^2 \geq \|g\|_2^2 > \sum_{k=1}^{2^r-1} \|g^r_{(r,k)}\|^2 \) we see that for at least \( 2^r-1 \) of the functions \( (g^r_{(r,k)})_{k=1}^{2^r-1} \) we have

\[
(6) \quad \|g^r_{(r,k)}\|^2 \leq 2C^2^{-r/2}.
\]

Using (5) for each \( k \) satisfying (6) we obtain a set \( B_k, \sigma(B_k) > 16^{-1}C^{-2} \), such that \( |g^r_{(r,k)}(\varsigma)| \geq 1/2 \cdot 2^{-r/2} \) for \( \varsigma \in B_k \).

From this follows that for some \( \alpha \) and \( \beta \) (independent of \( r \)) the set \( A_r \) of all points in \( S \), which belong to at least \( \alpha \cdot 2^r \) sets \( B_k \), has measure \( \geq \beta \). This clearly proves (4).

Using the “scrambling lemma” [3, p. 129] we find a sequence \( U_r \) of unitary maps such that almost every \( \varsigma \in S \) belongs to infinitely many sets \( U_r(A_r) \). We define the desired function by

\[
(7) \quad f = \sum_{r=1}^{\infty} r^{-2} g^r \circ U_r^{-1}.
\]

By (1) we infer that the homogeneous expansion \( \sum_{r=0}^{\infty} f_n \) for \( f \) actually equals \( \sum_{r=1}^{\infty} 2^{(r+1,0)} r^{-2} g^r \circ U_r^{-1} \). Since \( ||g^r \circ U_r^{-1}||_\infty = ||g^r||_\infty \leq C \) the series (7) is absolutely convergent and defines a function in \( A(B) \). By (4), for \( \varsigma \in U_r(A_r) \), we have \( \sum_k |g^r_k \circ U_r^{-1}(\varsigma)|^p \geq \alpha 2^r(1-p/2) \).

Given \( \varsigma \in S \)

\[
\sum_{n=0}^{\infty} |f_n(\varsigma)|^p = \sum_{r=1}^{\infty} r^{-(r+1,0)-1} \sum_{k=\gamma(r,0)+1}^{\gamma(r+1,0)} |g^r_k \circ U_r^{-1}(\varsigma)|^p \geq \sum_{r: \varsigma \in U_r(A_r)} \alpha r^{-2} 2^r(1-p/2).
\]

This sum is clearly infinite for every \( p < 2 \) and every \( \varsigma \) belonging to infinitely many sets \( U_r(A_r) \). This completes the proof.

**REMARK 1.** I do not know the answer to the following question: given \( (\alpha_n) \in \ell_2 \) does there exist \( f = \sum f_k \in A(B) \) such that \( f_{2^k} = \alpha_k p_{2^k} \)? A positive answer to this question would imply the theorem.

**REMARK 2.** Our proof of the proposition is the same as the one given for the disc algebra in [2, Proposition 3.1]. In one variable there are also beautiful constructive proofs (e.g. in [1]). It would be interesting to have such a proof in our case.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, 00-950 WARSAW, POLAND

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