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MEAN GROWTH AND SMOOTHNESS OF ANALYTIC FUNCTIONS

A. MATHESON

ABSTRACT. Let G_α denote the class of functions $f(z)$ analytic in the unit disk such that

$$\int_0^1 (1-r)^{-\alpha} M_\infty(f', r)\, dr < \infty,$$

for some α $(0 < \alpha < 1)$. A characterization of G_α is given in terms of moduli of continuity and an application is given to Riesz factorization of functions in G_α.

1. Let G_α, $0 < \alpha < 1$, denote the class of functions f analytic in the open unit disk U which satisfy

$$\psi_\alpha(f) = \int_0^1 (1-r)^{-\alpha} M_\infty(f', r)\, dr < \infty,$$

where $M_\infty(f', r) = \max\{|f'(re^{i\theta})|; 0 \le \theta \le 2\pi\}$. Defining $\|f\| = \|f\|_\infty + \psi_\alpha(f)$ yields a norm on G_α which is easily seen to make G_α into a Banach algebra. The purpose of this paper is to give a characterization of the functions of G_α in terms of the smoothness of their boundary functions. In the first place it is easily seen that the functions of G_α satisfy a Lipschitz condition of order α. In fact, somewhat more is true, as the following lemma shows. If f is analytic in U and continuous on $\overline{U}$, let $\omega(f, \delta)$ denote the modulus of continuity of f on the boundary of U.

LEMMA 1. *If $f \in G_\alpha$, then f has a continuous extension to $\overline{U}$, and $\omega(f, \delta) = o(\delta^\alpha)$.*

PROOF. By a well-known theorem of Hardy and Littlewood [**3**, p. 263], it suffices to show that $M_\infty(f', r) = o((1-r)^{\alpha-1})$. But $M_\infty(f', r)$ is nondecreasing, so this follows from the integrability condition.

The following theorem gives a precise condition on the modulus of continuity of a function for membership in the class G_α. The theorem will be proved through a series of lemmas in §2 and an application will be given in §3.

THEOREM 1. *Let f be analytic in U and continuous on $\overline{U}$. Then the following conditions are equivalent.*

(1) $\psi_\alpha(f) < \infty$,
(2) $\int_0^1 t^{-1-\alpha} \omega(f, t)\, dt < \infty$,
(3) $\sum_{n=1}^\infty 2^{n\alpha} \omega(f, 2^{-n}) < \infty$.

2. The first two lemmas of this section relate the modulus of continuity of a function f with the growth of its derivative. It would be surprising if these results

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were not known, but the proofs, which follow closely the proof of the theorem of Hardy and Littlewood cited above, will be included for completeness. First let $\omega(\delta)$ be an increasing continuous function for $0 \le \delta < \infty$, with $\omega(0) = 0$ and $\omega(\delta) = \omega(\pi)$ when $\delta \ge \pi$. Define

$$\tilde{\omega}(\delta) = \delta \int_\delta^\infty t^{-2}\omega(t)\,dt \quad \text{and} \quad \hat{\omega}(\delta) = \int_0^\delta t^{-1}\omega(t)\,dt.$$

Note that $\hat{\omega}(\delta) \equiv \infty$ may happen, but it is easy to see that $\omega(\delta) \le \tilde{\omega}(\delta)$ for all δ, and if $\omega(\delta) = o(\delta^\beta)$ for some β with $0 < \beta < 1$, then $\tilde{\omega}(\delta) = o(\delta^\beta)$ and $\hat{\omega}(\delta) = o(\delta^\beta)$. In particular, if $f \in G_\alpha$, then $\hat{\omega}(f,\delta)$ is finite.

LEMMA 2. *Let f be analytic in U and continuous on $\overline{U}$, and let $\omega(\delta)$ denote the modulus of continuity of f on the boundary of U. Then*

$$M_\infty(f', r) = O(\tilde{\omega}(1-r)/(1-r)) \quad \text{as } r \to 1^-.$$

PROOF. Fix a point $z = re^{it}$ in the unit disk U. Then

$$\begin{aligned} f'(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\frac{e^{i\theta}}{(e^{i\theta}-z)^2}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(e^{i\theta}) - f(e^{it})]\frac{e^{i\theta}}{(e^{i\theta}-z)^2}\,d\theta; \end{aligned}$$

so that

$$|f'(z)| \le \frac{1}{\pi}\int_0^\pi \frac{\omega(\delta)}{1 - 2r\cos\delta + r^2}\,d\delta.$$

Now if $\frac{1}{2} \le r < 1$, and if $|\delta| \le \pi$, then $1 - 2r\cos\delta + r^2 \ge (1-r)^2 + 2\delta^2/\pi^2$. Thus

$$\begin{aligned} |f'(z)| &\le \frac{1}{\pi}\int_0^\pi \frac{\omega(\delta)}{(1-r)^2 + 2\delta^2/\pi^2}\,d\delta \\ &\le \frac{1}{\pi}\int_0^{1-r} \frac{\omega(\delta)}{(1-r)^2}\,d\delta + \frac{\pi}{2}\int_{1-r}^{\pi}\frac{\omega(\delta)}{\delta^2}\,d\delta \\ &\le \frac{1}{\pi}\cdot\frac{\omega(1-r)}{1-r} + \frac{\pi}{2}\frac{\tilde{\omega}(1-r)}{1-r} \le \left(\frac{1}{\pi} + \frac{\pi}{2}\right)\frac{\tilde{\omega}(1-r)}{1-r}, \end{aligned}$$

since $\omega(1-r) \le \tilde{\omega}(1-r)$. This completes the proof.

LEMMA 3. *Let $f(z)$ be a function analytic in the unit disk U such that*

$$f'(z) = O(\phi(1-|z|)/(1-|z|)) \quad \text{as } |z| \to 1^-,$$

where $\phi(\delta)$ is a nondecreasing function such that $\phi(\delta) \to 0$ and $\hat{\phi}(\delta) \to 0$ as $\delta \to 0$. Then f can be extended to a continuous function on the closed unit disk $\overline{U}$ such that $\omega(f,\delta) = O(\phi(\delta) + \hat{\phi}(\delta))$.

PROOF. Since $\hat{\phi}(\delta) \to 0$ as $\delta \to 0$, $\delta^{-1}\phi(\delta)$ is integrable on $0 \le \delta \le 1$, and so $f'(z)$ is integrable over any radius of the unit disk. In particular $f(z)$ has finite radial limits everywhere on the unit circle.

Now let θ and θ' be two real numbers such that $0 < \theta' - \theta < 1$ and choose r so that $1 - r = \theta' - \theta$. Let C_1 be the radial segment from $e^{i\theta}$ to $re^{i\theta}$, C_2 the circular arc from $re^{i\theta}$ to $re^{i\theta'}$, and C_3 the radial segment from $re^{i\theta'}$ to $e^{i\theta'}$. If C is the contour consisting of C_1 followed by C_2 followed by C_3, then $f(e^{i\theta'}) - f(e^{i\theta}) = \int_C f'(z)\,dz$. Thus to estimate $|f(e^{i\theta'}) - f(e^{i\theta})|$ it suffices to estimate the integral of $f'(z)$ separately over the contours C_1, C_2 and C_3. On C_1,

$$\left|\int_{C_1} f'(z)\,dz\right| \leq \int_{C_1} |f'(z)||dz| \leq C\int_r^1 \frac{\phi(1-s)}{1-s}\,ds = C\hat{\phi}(1-r).$$

Similarly, $|\int_{C_3} f'(z)\,dz| \leq C\hat{\phi}(1-r)$. On C_2,

$$\left|\int_{C_2} f'(z)\,dz\right| \leq C\frac{\phi(1-r)}{1-r}|\theta-\theta'| = C\phi(1-r).$$

Combining these estimates completes the proof.

LEMMA 4. *Let f be continuous on $\overline{U}$ and analytic in U. Then*

$$\int_0^1 (1-r)^{-\alpha} M_\infty(f', r)\,dr \leq C_1\int_0^1 t^{-1-\alpha}\omega(f,t)\,dt + C_2\omega(f,\pi),$$

for some constants C_1 and C_2 depending only on α.

PROOF. Let $\omega(t) = \omega(f,t)$. By Lemma 2, $M_\infty(f',r) \leq C_0\tilde{\omega}(1-r)/(1-r)$ for some absolute C_0. Thus

$$\begin{aligned}
\int_0^1 (1-r)^{-\alpha} M_\infty(f',r)\,dr &\leq C_0\int_0^1 (1-r)^{-\alpha-1}\tilde{\omega}(1-r)\,dr \\
&= C_0\int_0^1 t^{-\alpha-1}\tilde{\omega}(t)\,dt \\
&= C_0\int_0^1 t^{-\alpha}\int_t^\infty s^{-2}\omega(s)\,ds \\
&= C_0\int_0^\infty\int_0^{s\wedge 1} t^{-\alpha}\,dt s^{-2}\omega(s)\,ds \\
&= \frac{C_0}{1-\alpha}\int_0^\infty (s\wedge 1)^{1-\alpha}s^{-2}\omega(s)\,ds \\
&= \frac{C_0}{1-\alpha}\left[\int_0^1 s^{-\alpha-1}\omega(s)\,ds + \int_1^\infty s^{-2}\omega(s)\,ds\right] \\
&= \frac{C_0}{1-\alpha}\left[\int_0^1 s^{-\alpha-1}\omega(s)\,ds + \omega(\pi)\right].
\end{aligned}$$

Here $s\wedge 1 = \min(s,1)$. This proves the lemma.

LEMMA 5. *Let ω be a nondecreasing function for $0 \leq t \leq 1$. Then there exist positive constants c and C depending only on α such that*

$$c\sum_{n=0}^\infty 2^{n\alpha}\omega(2^{-n}) \leq \int_0^1 t^{-1-\alpha}\omega(t)\,dt \leq C\sum_{n=0}^\infty 2^{n\alpha}\omega(2^{-n}).$$

PROOF. Evidently

$$\frac{1-2^{-\alpha}}{\alpha}2^{n\alpha}\omega(2^{-n}) \leq \int_{2^{-n}}^{2^{-n+1}} t^{-1-\alpha}\omega(t)\,dt \leq \frac{2^\alpha-1}{\alpha}2^{(n-1)\alpha}\omega(2^{-n+1}),$$

and the lemma follows on summation.

LEMMA 6. *If $f \in G_\alpha$ then $\int_0^1 t^{-1-\alpha}\omega(f,t)\,dt < \infty$.*

PROOF. For $n = 1, 2, \ldots$, choose t_n and t'_n so that $|t_n - t'_n| = 2^{-n}$ and $\omega(2^{-n}) = |f(e^{it_n}) - f(e^{it'_n})|$. Let $r_n = 1 - 2^{-n}$ and let C_n be the contour from e^{it_n} to $e^{it'_n}$ constructed as in the proof of Lemma 3. Since

$$f(e^{it'_n}) - f(e^{it_n}) = \int_{C_n} f'(z)\,dz,$$

it follows that

$$\begin{aligned}\omega(2^{-n}) = |f(e^{it'_n}) - f(e^{it_n})| &\le \int_{C_n} |f'(z)||dz| \\ &\le 2\int_{r_n}^1 M_\infty(f', r)\,dr + 2^{-n}M_\infty(f', r_n).\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} 2^{n\alpha}\omega(2^{-n}) \le 2\sum_{n=0}^{\infty} 2^{n\alpha}\int_{r_n}^1 M_\infty(f', r)\,dr + \sum_{n=0}^{\infty} 2^{n(\alpha-1)}M_\infty(f', r_n).$$

But

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{n\alpha}\int_{r_n}^1 M_\infty(f', r)\,dr &= \sum_{n=0}^{\infty} 2^{n\alpha}\sum_{k=n}^{\infty}\int_{r_k}^{r_{k+1}} M_\infty(f', r)\,dr \\ &= \sum_{k=0}^{\infty}\sum_{n=0}^{k} 2^{n\alpha}\int_{r_k}^{r_{k+1}} M_\infty(f', r)\,dr \\ &\le \frac{2^\alpha}{2^\alpha - 1}\sum_{k=0}^{\infty} 2^{k\alpha}\int_{r_k}^{r_{k+1}} M_\infty(f', r)\,dr \\ &\le \frac{2^\alpha}{2^\alpha - 1}\sum_{k=0}^{\infty}\int_{r_k}^{r_{k+1}} (1-r)^{-\alpha}M_\infty(f', r)\,dr \\ &= \frac{2^\alpha}{2^\alpha - 1}\int_0^1 (1-r)^{-\alpha}M_\infty(f', r)\,dr,\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{n(\alpha-1)}M_\infty(f', r_n) &\le \sum_{n=0}^{\infty}\int_{r_n}^{r_{n+1}} (1-r)^{-\alpha}M_\infty(f', r)\,dr \\ &= \int_0^1 (1-r)^{-\alpha}M_\infty(f', r)\,dr.\end{aligned}$$

Thus the lemma follows from Lemma 5.

The proof of Theorem 1 is now immediate. The equivalence of (2) and (3) follows from Lemma 5. Lemma 6 is the implication (1) implies (2), and Lemma 4 guarantees that (2) implies (1).

3. As an application of the above analysis, this section describes certain factorization properties of the spaces G_α. The next theorem is the basic result.

THEOREM 2. *Let $s \in H^\infty$ and $f \in G_\alpha$, and define*

$$T_s f(z) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)\overline{s(w)}}{w-z}\,dw.$$

Then $T_s f \in G_\alpha$, and there is a constant C depending only on α such that $\|T_s f\| \le C\|s\|_\infty \|f\|$.

PROOF. For $z \in U$, let $\hat{z} = z|z|^{-1}$. Since

$$(T_s f)'(z) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f(w)\overline{s(w)}}{(w-z)^2}\,dw = \frac{1}{2\pi i}\int_{|w|=1} \frac{[f(w)-f(\hat{z})]\overline{s(w)}}{(w-z)^2}\,dw,$$

it follows that

$$|(T_s f)'(z)| \le \frac{\|s\|_\infty}{2\pi}\int_{|w|=1} \frac{|f(w)-f(\hat{z})|}{|w-z|^2}\,|dw|.$$

Thus, using the argument of Lemma 2,

$$M_\infty((T_s f)', r) \le C\|s\|_\infty \frac{\tilde{\omega}(f, 1-r)}{1-r},$$

and the theorem follows as in the proof of Lemma 4.

Every function $f \in H^\infty$ can be factored as the product of an inner function and an outer function, and an inner function G is said to divide f if $fG^{-1} \in H^\infty$.

COROLLARY 1. *If $f \in G_\alpha$ and G is an inner function dividing f, then $fG^{-1} \in G_\alpha$, and $\|fG^{-1}\| \le C\|f\|$, where C is the constant of Theorem 2.*

PROOF. If $fG^{-1} \in H_\infty$, then fG^{-1} has boundary values $f(e^{i\theta})\overline{G(e^{i\theta})}$, and $fG^{-1} = T_G f$.

COROLLARY 2. *If $f \in G_\alpha$ and $f = GF$, where G is inner and F is outer, then $F \in G_\alpha$ and $\|F\| \le C\|f\|$, where C is the constant of Theorem 2.*

Corollary 2 is analogous to a result of Šamojan [**2**], who proves, essentially, that if f has modulus of continuity $\omega(\delta)$, then $\omega(F, \delta) = O(\omega(\delta) + \tilde{\omega}(\delta))$, where F is the outer factor of f. This should be compared with Theorem III, 13.30 of Zygmund [**3**, p. 121], and also the paper of Havin [**1**].

4. In conclusion several remarks are in order.

(1) The above analysis can be extended to integral moduli of continuity $\omega_p(f, \delta)$ $(1 \le p < \infty)$ and to higher derivative. It should be noted that

$$\int_0^1 (1-r)^{-1} M_\infty(f', r)\,dr < \infty$$

implies that f is constant.

(2) If u is continuous on $\overline{U}$ and harmonic in U, and satisfies $\int_0^1 t^{-1-\alpha}\omega(u, t)\,dt < \infty$, then the techniques above can be used to show that the modulus of continuity of its conjugate function $\tilde{u}$ satisfies the same condition. In particular the functions of G_α are characterized by their real parts.

(3) Since G_α is a Banach algebra of analytic functions, it is natural to ask for a description of its closed ideals. The description, which parallels the description for Lipschitz spaces of analytic functions, will be presented in a forthcoming paper.

5. Part of the revision of this work was accomplished while the author was visiting the University of North Carolina at Chapel Hill.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078