A UNIQUENESS RESULT IN CONFORMAL MAPPING. II

JAMES A. JENKINS

ABSTRACT. This paper gives an elementary proof of the result that for a function \( f \) in the family \( \Sigma \) the diameter of the complement of the image of \( |z| > 1 \) by \( w = f(z) \) attains its minimal value 2 only for \( f(z) = z + c \), \( c \) constant.

1. Some years ago the author gave the first published proof [2] of the result that, for a function \( f \) in the family \( \Sigma \) [1], the diameter of the complement of the image of \( |z| > 1 \) by \( w = f(z) \) attains its minimal value 2, then \( f(z) = z + c \) (\( c \) constant). While the proof is very simple, it utilizes results and concepts which would not normally be familiar to a student in a good basic course in Function Theory, which is the natural context for the above result. More recently Pfluger [3, 4] has given several versions of a proof of this result which seem to him to be more elementary. It appears that they also would require some digression from the material usually found in a Function Theory course. Our purpose here is to give an elementary proof in purely Function Theoretic terms.

2. LEMMA 1. Let \( f \in \Sigma \) have Laurent expansion about the point at infinity

\[
f(z) = z + c_0 + \sum_{n=1}^{\infty} c_n z^{-n}.
\]

Let the complement of the image of \( |z| > 1 \) under \( w = f(z) \) have diameter \( D \). Then \( D \geq 2 \) and a necessary condition for equality is \( c_{2n+1} = 0 \), \( n \) integral \( \geq 0 \).

Let \( D_r \) be the diameter of the image of \( |z| = r \), \( r > 1 \), under \( w = f(z) \); \( D_r^* = \max_{|z|=r} |f(z) - f(-z)| \). Evidently \( D_r \geq D_r^* \geq D \) and \( \lim_{r \to 1} D_r = D \). Further

\[
(D_r^*)^2 \geq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(-re^{i\theta})|^2 d\theta = 4r^2 + 4 \sum_{n=0}^{\infty} |c_{2n+1}|^2 r^{-2(2n+1)}.
\]

Thus \( D \geq 2 \) with equality only if \( c_{2n+1} = 0 \), \( n = 0, 1, 2, \ldots \).

The following lemma is found in [3].

LEMMA 2. Using the same notation as in Lemma 1, \( D_r \leq rD \), \( r > 1 \).

Let \( |z_1| = |z_2| = r \), \( r > 1 \), be such that \( |f(z_1) - f(z_2)| = D_r \) and let \( \Phi(z) = f(z_1 z_2^{-1} z) - f(z) \). Then for \( \rho > 1 \), \( \max_{|z|=\rho} |\Phi(z)| \leq D_\rho \). The function \( z^{-1} \Phi(z) \) is regular outside the unit circle (including the point at infinity) thus \( |z^{-1} \Phi(z)| \leq \rho^{-1} D_\rho \) in \( |z| \geq \rho \). Letting \( \rho \) tend to 1 we have \( |z^{-1} \Phi(z)| \leq D \) in \( |z| > 1 \). Thus \( D_r \leq rD \).

Received by the editors May 14, 1981.
1980 Mathematics Subject Classification. Primary 30C25, 30C55, 30C75.
1Research supported in part by the National Science Foundation.

© 1982 American Mathematical Society
0002-9939/81/0000-0300/$01.50
THEOREM. Let \( f \in \Sigma \) and let the complement of the image of \(|z| > 1\) under \( w = f(z) \) have diameter \( D \). Then \( D \geq 2 \) and \( D = 2 \) if and only if \( f(z) = z + c \), \( c \) constant.

It remains only to consider the possibility \( D = 2 \). If \( f(z) \) is not \( z + c \), let it have the Laurent expansion (1). By Lemma 1 the first nonzero coefficient beyond the constant term would be \( c_{2n} \), \( n \geq 1 \). By Lemmas 1 and 2, \( D_r = D_{r^*} = 2r \) thus \( |f(re^{i\theta}) - f(re^{i\omega})|^2 \) would be maximized for \( \varphi = \theta + \pi \) for all \( r > 1 \), \( \theta \) real and the partial derivative of this quantity with respect to \( \varphi \) evaluated at \( \varphi = \theta + \pi \) would be identically zero in this set. Inserting the expansion (1) for \( f \) the coefficient of \( r^{-2n+1} \) would be

\[
4ni(c_{2n}e^{-(2n+1)i\theta} - \bar{c}_{2n}e^{(2n+1)i\theta})
\]

which would imply the contradiction \( c_{2n} = 0 \).

REFERENCES


SCHOOL OF MATHEMATICS, THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address: Department of Mathematics, Washington University, St. Louis, Missouri 63130