HILBERTIAN INTERPRETATIONS OF MANUALS

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ABSTRACT. We characterize manuals which admit an interpretation in a manual on a Hilbert space. This characterization is given in terms of a certain set of states that the manual supports.

1. Introduction. In a series of papers [1–4, 8–10], D. Foulis and C. Randall developed a mathematical formalism for quantum mechanics and other empirical sciences. Their formalism is at a more primitive level than the quantum logic approach [5, 6, 7, 11] and, in fact, the latter can be derived from the former. In their approach, the physical operations form the basis of an axiomatic system in which the operations band together to form a mathematical structure called a manual.

A basic problem in axiomatic quantum mechanics is to characterize general quantum systems which are isomorphic to the traditional Hilbert space quantum mechanics. In this paper we give such a result for manuals. In particular, we show that a manual $A$ admits a Hilbert space morphism if and only if $A$ supports a collection of states of a certain type.

2. Definitions and notation. Most of the definitions in this section may be found in the work of Foulis and Randall cited in the introduction. Let $X$ be a nonempty set and let $A = A(X)$ be a collection of nonempty subsets of $X$ such that $X = \bigcup A$. The elements of $X$ are called outcomes, and the sets in $A$ are called operations. Any subset of an operation is an event. Denote the set of events by $\mathcal{E}(A)$. If $x, y \in E \in A$ and $x \neq y$ we write $x \perp y$. If $A \subseteq X$ we write

$$A \perp = \{x \in X: x \perp y \forall y \in A\}$$

and for $A, B \subseteq X$ we write $A \perp B$ if $A \subseteq B \perp$. We call $A$ a manual if

(1) $E, F \in A$ and $E \subseteq F$ implies $E = F$;
(2) $A, B \in \mathcal{E}(A)$ and $A \perp B$ implies $A \cup B \in \mathcal{E}(A)$.

A morphism $\phi$ from a manual $A$ to a manual $B$ is a map $\phi: \mathcal{E}(A) \rightarrow \mathcal{E}(B)$ such that

(1) if $A_i \in \mathcal{E}(A)$ and $A = \bigcup A_i \in \mathcal{E}(A)$, then $\phi(A) = \bigcup \phi(A_i)$;
(2) if $A, B \in \mathcal{E}(A)$ and $A \perp B \subseteq B \perp$, then $\phi(A) \perp \subseteq \phi(B) \perp$.

Let $\phi$ be a morphism from $A(X)$ to $B(Y)$. Then $\phi$ is said to be

(a) outcome preserving if $x \in X$ implies $\phi(x) \in Y$;
(b) operation preserving if $E \in A$ implies $\phi(E) \in B$;
(c) faithful if $A, B \in \mathcal{E}(A)$ and $\phi(A) \perp \phi(B)$ implies $A \perp B$;
(d) conditioning if $A, B \in \mathcal{E}(A)$ and $A \perp B$ implies $\phi(A) \perp \phi(B)$;

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(e) injective if \( \phi \) is outcome preserving and \( \phi(x) = \phi(y) \) implies \( x = y \);
(f) a homomorphism if \{\( \phi(E) \colon E \in \mathcal{A} \}\} is a manual;
(g) a homomorphism onto if \( \mathcal{B} = \{\phi(E) \colon E \in \mathcal{A} \}\);
(h) an isomorphism if \( \phi \) is injective and a homomorphism onto \( \mathcal{B} \).

An operation preserving, conditioning morphism is an interpretation.

A state for a manual \( \mathcal{A} \) is a map \( \alpha \colon \mathcal{E}(\mathcal{A}) \to [0,1] \) such that

1. if \( E \in \mathcal{A} \) then \( \alpha(E) = 1 \); 
2. if \( A, B \in \mathcal{E}(\mathcal{A}) \) and \( A \perp B \), then \( \alpha(A \cup B) = \alpha(A) + \alpha(B) \).

A state \( \alpha \) for \( \mathcal{A} \) is regular if for any family \( A_i \in \mathcal{E}(\mathcal{A}) \) with \( A_i \perp A_j \), \( i \neq j \), such that \( A = \bigcup A_i \in \mathcal{E}(\mathcal{A}) \) we have \( \alpha(A) = \sum \alpha(A_i) \). We denote the set of regular states on \( \mathcal{A} \) by \( \sum(\mathcal{A}) \). If \( \alpha \in \sum(\mathcal{A}) \), \( x \in \mathcal{X} \), we write \( \alpha(x) = \alpha(\{x\}) \). It is clear that the function \( x \to \alpha(x) \) determines \( \alpha \). A function \( f \colon \mathcal{X} \to \mathbb{C} \) is an amplitude function \([12]\) for \( \alpha \in \sum(\mathcal{A}) \) if \( |f(x)|^2 = \alpha(x) \) for all \( x \in \mathcal{X} \). Clearly, any \( \alpha \in \sum(\mathcal{A}) \) has many amplitude functions. Indeed, \( f(x) = \lambda \alpha(x)^{1/2} \) is such a function for any \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \). We call a function \( f \colon \mathcal{X} \to \mathbb{C} \) summable if \( \sum_{x \in \mathcal{X}} f(x) \) exists for all \( E \in \mathcal{A} \) and \( \sum_{x \in \mathcal{F}} f(x) = \sum_{x \in \mathcal{E}} f(x) \) for all \( E, F \in \mathcal{A} \). An example of a summable function is \( x \to \alpha(x) \) for any \( \alpha \in \sum(\mathcal{A}) \).

We now give an example of a manual which is of importance to our present work. Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathcal{S}(\mathcal{H}) = \{v \in \mathcal{H} : ||v|| = 1\} \) be its unit sphere. Let \( \mathcal{H}_1 \) be the set of all one-dimensional (orthogonal) projections on \( \mathcal{H} \) and let \( \mathcal{A}(\mathcal{H}_1) \) be the collection of all maximal orthogonal sets in \( \mathcal{H}_1 \). Then \( \mathcal{A}(\mathcal{H}_1) \) is a manual. We call \( \mathcal{A}(\mathcal{H}_1) \) the Hilbertian manual on \( \mathcal{H} \). Our Hilbertian manual is a submanual of the Foulis-Randall Hilbert space manual \([4]\) consisting of the collection of all maximal orthogonal sets of projections on \( \mathcal{H} \). Both these manuals generate the same “operational logic” \([4]\). It is frequently convenient to consider the elements of \( \mathcal{H}_1 \) as one-dimensional subspaces of \( \mathcal{H} \).

What distinguishes a Hilbertian manual \( \mathcal{A}(\mathcal{H}_1) \) from among others in the general class of manuals? The Hilbertian manuals support a special set of states. Corresponding to a \( v \in \mathcal{S}(\mathcal{H}) \) we define the vector state \( \hat{v} \) by \( \hat{v}(p) = \langle pv, v \rangle \) for all \( p \in \mathcal{H}_1 \). Let \( \mathcal{V}(\mathcal{H}) = \{\hat{v} : v \in \mathcal{S}(\mathcal{H})\} \). The set of vector states \( \mathcal{V}(\mathcal{H}) \) has two important properties. First, if \( \emptyset \neq A \in \mathcal{E}(\mathcal{A}) \), then there exists a \( \hat{v} \in \mathcal{V}(\mathcal{H}) \) such that \( \hat{v}(A) = 1 \). For the second property, for each \( p \in \mathcal{H}_1 \) choose a \( \hat{p} \in \mathcal{S}(\mathcal{H}) \) such that \( \hat{p} \perp p \). If \( E \in \mathcal{A}(\mathcal{H}_1) \), it is clear that \( \{\hat{p} : p \in E\} \) is an orthonormal basis for \( \mathcal{H} \). Now for each \( \hat{v} \in \mathcal{V}(\mathcal{H}) \) define the amplitude function \( f_{\hat{v}} : \mathcal{H}_1 \to \mathbb{C} \) by \( f_{\hat{v}}(p) = \langle v, \hat{p} \rangle \). Then for any \( \hat{u}, \hat{v} \in \mathcal{V}(\mathcal{H}) \), the function \( f_{\hat{u}}f_{\hat{v}} \) is summable. Indeed, if \( E, F \in \mathcal{A}(\mathcal{H}_1) \) we have

\[
\sum_{p \in E} f_{\hat{u}}(p)f_{\hat{v}}(p) = \sum_{p \in E} \langle u, \hat{p} \rangle \langle \hat{p}, v \rangle = \langle u, v \rangle = \sum_{p \in F} \langle u, \hat{p} \rangle \langle \hat{p}, v \rangle = \sum_{p \in F} f_{\hat{u}}(p)f_{\hat{v}}(p).
\]

We close this section by defining families of states with certain special properties. Let \( \mathcal{A}(\mathcal{X}) \) be a manual and let \( \Delta \subseteq \sum(\mathcal{A}) \). Then \( \Delta \) is said to be

1. unital if for any \( \emptyset \neq A \in \mathcal{E}(\mathcal{A}) \) there exists an \( \alpha \in \Delta \) such that \( \alpha(A) = 1 \);
2. strongly separating if \( x, y \in \mathcal{X} \) and \( x \neq y \) implies that there exists an \( \alpha \in \Delta \) such that \( \alpha(x) \neq \alpha(y) \);
3. strongly \perp-determining if \( \{\alpha \in \Delta : \alpha(A) = 1\} \subseteq \{\alpha \in \Delta : \alpha(B) = 0\} \) implies \( A \perp B \), \( A, B \in \mathcal{E}(\mathcal{A}) \);
4. a set of amplitude states if for every \( \alpha \in \Delta \) there is an amplitude function \( f_\alpha \) for \( \alpha \) such that for every \( \alpha, \beta \in \Delta \), \( f_\alpha f_\beta \) is summable.
We have shown above that $\mathcal{V}(H)$ is a unital set of amplitude states. It is also straightforward to show that $\mathcal{V}(H)$ is strongly separating and strongly $\perp$-determining.

3. Hilbertian interpretations. We now characterize those manuals which have a Hilbertian interpretation.

**Theorem.** A manual $A(X)$ has an outcome preserving interpretation $\phi$ in a Hilbertian manual $A(H)$ if and only if $A(X)$ admits a unital set of amplitude states $\Delta$. Moreover, if the above condition holds, then there exists an injection $\psi: \Delta \to \mathcal{V}(H)$ such that $\alpha(A) = \psi(\alpha)(\phi(A))$ for all $A \in \mathcal{E}(A)$.

**Proof.** Suppose $\phi: \mathcal{E}(A) \to \mathcal{E}[A(H)]$ is an outcome preserving interpretation. Then for every $A \in \mathcal{E}(A)$, $\phi(A) = \bigcup_{a \in A} \phi(a)$. For each $\phi \in \mathcal{V}(H)$, $A \in \mathcal{E}(A)$, define $\alpha_{\phi}(A) = \phi(\phi(A)) = \sum_{a \in A} \phi(a)$. It is straightforward to show that each $\alpha_{\phi}$ is a regular state on $A(X)$. Let $\Delta = \{\alpha_{\phi} : \phi \in \mathcal{V}(H)\}$. To show that $\Delta$ is unital, let $\emptyset \neq A \in \mathcal{E}(A)$ and let $p \in \phi(A)$. Then there exists a $\phi \in S(H)$ such that $\phi \in p$. Hence, $\alpha_{\phi}(A) = \phi(\phi(A)) = 1$. To show that $\Delta$ is a set of amplitude states, for each $p \in H_1$ choose a $\tilde{p} \in S(H)$ such that $\tilde{p} \in p$. For $\alpha_{\phi} \in \Delta$, define $f_{\alpha_{\phi}}: X \to C$ by $f_{\alpha_{\phi}}(x) = \langle \phi(x), \phi(x) \rangle$. Then $f_{\alpha_{\phi}}$ is a unital function on $A(X)$.

Finally, for any $E, F \in A(X)$, $\alpha_{u}, \alpha_{v} \in \Delta$ we have

$$\sum_{x \in E} f_{u}(x) f_{v}(x) = \sum_{x \in E} \langle u, \phi(x) \rangle \langle \phi(x), v \rangle = \langle u, v \rangle$$

Conversely, let $\Delta$ be a unital set of amplitude states on $A(X)$ and let $H_0 = \{f_{\alpha} : \alpha \in \Delta\}$ be the corresponding set of amplitude functions. Let $[H_0]$ be the linear span of $H_0$. For $f, g \in [H_0]$ define $\langle f, g \rangle = \sum_{x \in X} f(x) g(x)$ where $E \in A$. It is straightforward to show that $\langle f, g \rangle$ is well defined and independent of $E \in A$. It is clear that $\langle \cdot, \cdot \rangle$ is an inner product. Let $H$ be the Hilbert space completion of $[H_0]$ relative to the inner product $\langle \cdot, \cdot \rangle$. We now show that if $f \in H$, then $f: X \to C$ and $\|f\|^2 = \sum_{x \in X} |f(x)|^2 = \sum_{x \in X} |f(x)|^2$ for every $E, F \in A$. Moreover, if $f, g \in H$, then $\langle f, g \rangle = \sum_{x \in X} f(x) g(x) = \sum_{x \in X} f(x) g(x)$ for every $E, F \in A$. Indeed, let $f_i$ be a Cauchy sequence in $[H_0]$. Then $f_i(x)$ converges in $C$ for every $x \in X$ and hence there exists an $f: X \to C$ such that $f_i(x) \to f(x)$ for all $x \in X$. It follows that $\|f - f_i\| \to 0$ as $i \to \infty$. Also,

$$\|f\|^2 = \lim_{i \to \infty} \|f_i\|^2 = \lim_{i \to \infty} \sum_{x \in E} |f_i(x)|^2 = \lim_{i \to \infty} \sum_{x \in F} |f_i(x)|^2$$

The second statement now follows from the polarization identity.

For $x \in X$, let $\phi(x)$ be the linear span $\{f_{\alpha} : \alpha(x) = 1, \alpha \in \Delta\} \subseteq H$. We now show that $\phi(x)$ is one-dimensional and hence in $H_1$. Suppose $\alpha(x) = \beta(x) = 1$ for $\alpha, \beta \in \Delta$. If $y \perp x$, then $\alpha(y) = \beta(y) = 0$ and hence, $f_{\alpha}(y) = f_{\beta}(y) = 0$. Thus, if
Since we have an equality in Schwarz's inequality, there is a \( \lambda \in \mathbb{C} \) such that \( f_\beta = \lambda f_\alpha \). Hence, \( \phi: X \to H_1 \) and \( \phi \) preserves outcomes. Extend \( \phi \) to \( \mathcal{E}(A) \) by defining \( \phi(A) = \bigcup_{x \in A} \phi(x) \) for \( A \in \mathcal{E}(A) \). If \( E \in A \), we now show that \( \phi(E) \in \mathcal{A}(H_1) \). Let \( x, y \in E \) with \( x \neq y \) and suppose \( \alpha(x) = \beta(y) = 1 \), \( \alpha, \beta \in \Delta \). Then

\[
\langle f_\alpha, f_\beta \rangle = \sum_{x \in E} f_\alpha(x) \overline{f_\beta(y)} = \sum_{x \in E} \delta_{x,y} = 0.
\]

Hence, \( \phi(x) \perp \phi(y) \). Now let \( p \in H_1 \) satisfy \( p \perp \phi(E) \). If \( g \in p \), \( x \in E \) and \( \alpha(x) = 1 \), then \( g \perp f_\alpha \). Hence,

\[
0 = \langle g, f_\alpha \rangle = \sum_{y \in E} g(y) \overline{f_\alpha(y)} = g(x) \overline{f_\alpha(x)}.
\]

Hence, \( g(x) = 0 \) for all \( x \in E \) and \( \|g\|^2 = \sum_{x \in E} |g(x)|^2 = 0 \). Thus, \( g = 0 \). It follows that \( \phi(E) \in \mathcal{A}(H_1) \) so \( \phi \) preserves operations and events. Hence, \( \phi: \mathcal{E}(A) \to \mathcal{E}(\mathcal{A}(H_1)) \). Moreover, if \( x \perp y \) then \( \phi(x) \perp \phi(y) \) so \( A, B \in \mathcal{E}(A) \) and \( A \perp B \) implies \( \phi(A) \perp \phi(B) \).

We now show that \( \phi \) is a morphism and therefore an interpretation (actually this follows from [4, Lemma 2], but we shall give the proof to make this work self-contained). First, condition (1) in the definition of a morphism clearly holds. For condition (2), suppose \( A, B \in \mathcal{E}(A) \) and \( A \perp B \subseteq B \perp \). Let \( p \in \phi(A) \perp \) and \( q \in \phi(B) \) and let \( f, g \in \mathcal{H} \) satisfy \( f \in p \), \( g \in q \). Let \( E \in A \) satisfy \( x \subseteq E \). Then \( E - A \subseteq A \perp \subseteq B \perp \) and hence

\[
\phi(E - A) \subseteq \phi(B \perp) \subseteq \phi(B) \perp \subseteq q \perp.
\]

Hence \( q \in \phi(E - A) \perp \). Since \( \sum_{x \in E} \phi(x) = I \), we have \( f = |\sum_{x \in E - A} \phi(x)| f \) and \( g = |\sum_{x \in A} \phi(x)| g \). Hence, \( f \perp g \) so \( p \perp g \) and \( \phi(A) \perp \subseteq \phi(B) \perp \).

For the last statement of the theorem, define \( \psi: \Delta \to \mathcal{V}(H) \) by \( \psi(\alpha) = \tilde{\alpha} \). Let \( x \in X \) and let \( \beta \in \Delta \) satisfy \( \beta(x) = 1 \). Then for any \( \alpha \in \Delta \) we have

\[
\psi(\alpha)[\phi(x)] = \tilde{\alpha}[\phi(x)] = \langle \phi(x)f_\alpha, f_\alpha \rangle = |(f_\alpha, f_\beta)|^2 = |f_\alpha(x)|^2 = \alpha(x).
\]

Hence, for any \( A \in \mathcal{E}(A) \) we have

\[
\psi(\alpha)[\phi(A)] = \psi(\alpha) \sum_{x \in A} \phi(x) = \sum_{x \in A} \psi(\alpha)[\phi(x)] = \sum_{x \in A} \alpha(x) = \alpha(A). \]

**COROLLARY.** A manual \( \mathcal{A}(X) \) has an injective interpretation in a Hilbertian manual if and only if \( \mathcal{A}(X) \) admits a unital, strongly separating set of amplitude states.

**PROOF.** Suppose \( \Delta \) is a unital, strongly separating set of amplitude states on \( \mathcal{A}(X) \). Let \( \phi: X \to H_1 \) and \( \psi: \Delta \to \mathcal{V}(H) \) be the maps constructed in the theorem. To show that \( \phi \) is injective assume that \( \phi(x) = \phi(y) \). Then for any \( \alpha \in \Delta \) we have

\[
\alpha(y) = \psi(\alpha)[\phi(y)] = \psi(\alpha)[\phi(x)] = \alpha(x).
\]
Hence $x = y$. Conversely, suppose $\phi: \mathcal{E}(A) \to \mathcal{E}[A(H_1)]$ is an injective interpretation in a Hilbertian manual $A(H_1)$. Define $\Delta = \{\alpha_\nu : \nu \in \mathcal{V}(H)\}$ as in the theorem. The theorem shows that $\Delta$ is a unital set of amplitude states on $A$. To show that $\Delta$ is strongly separating assume that $\alpha(x) = \alpha(y)$ for all $\alpha \in \Delta$. Then $\hat{\nu}[\phi(x)] = \hat{\nu}[\phi(y)]$ for all $\nu \in \mathcal{V}(H)$. Since $\mathcal{V}(H)$ is strongly separating on $A(H_1)$ we have $\phi(x) = \phi(y)$. Since $\phi$ is injective, $x = y$. □

**COROLLARY.** A manual $A$ is isomorphic to a submanual of a Hilbertian manual if and only if $A$ admits a unital, strongly separating set of amplitude states.

**COROLLARY.** A manual $A$ has an outcome preserving faithful interpretation in a Hilbertian manual if and only if $A$ admits a unital, strongly $\bot$-determining set of amplitude states.

**COROLLARY.** If a manual $A$ admits a unital set of amplitude states $\Delta$, then $A$ has an outcome preserving interpretation in a Hilbert space manual.

Corresponding results hold in the last corollary if $\Delta$ is separating or strongly $\bot$-determining.

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