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A KRASNOSEL'SKII-TYPE THEOREM FOR POINTS OF LOCAL NONCONVEXITY

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ABSTRACT. Let S be a compact connected set in R^2, S not convex. Then S is starshaped if and only if every 3 points of local nonconvexity of S are clearly visible from a common point of S. For $k = 1$ or $k = 2$, $\dim \ker S \geq k$ if and only if for some $\epsilon > 0$, every $f(k) = \max\{3, 6 - 2k\}$ points of local nonconvexity of S are clearly visible from a common k-dimensional ϵ-neighborhood in S. Each result is best possible.

1. Introduction. We begin with some preliminary definitions. A point s in S is said to be a *point of local nonconvexity* of S if and only if there is some neighborhood N of s such that $S \cap N$ is convex. In case S fails to be locally convex at point q in S, then q is called a *point of local nonconvexity* (lnc point) of S. For points x and y in S, we say x *sees* y *via* S (x is *visible* from y via S) if and only if the corresponding segment $[x, y]$ lies in S. Point x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees each point of $S \cap N$ via S. Set S is called starshaped if and only if there is some point p in S such that p sees each point of S via S, and the set of all such points p is called the (convex) kernel of S, denoted $\ker S$.

A well-known theorem of Krasnosel'skii [6] states that if S is a nonempty compact set in R^d, then S is starshaped if and only if every $d+1$ points of S see a common point via S. Moreover, results in [1] indicate a relationship between the kernel of S and the set of lnc points of S: It is proved that for S a nonempty compact subset of R^2 having n lnc points, then the kernel of S contains an interval of radius ϵ if and only if every $f(n) = \max\{4, 2n\}$ points of S see such an interval via S. Thus it seems natural to expect that the set Q of lnc points of S might be used to obtain a Krasnosel'skii-type theorem for starshaped sets, independent of the cardinality of Q. Here we use the concept of clearly visible, a notion which appears in a paper by Stavrakas [7] and in work by Falconer [4], to obtain the following result: For S compact, connected, and nonempty in R^2, S is starshaped if and only if every 3 points of local nonconvexity of S are clearly visible from a common point of S. Furthermore, this result is used to produce analogues of theorems in [3] which reveal the dimension of the kernel of S. Notice that this suggests a method of generating analogues to existing theorems about starshaped sets: Suppose a familiar theorem states that the kernel of some compact starshaped set S has a particular configuration if and only if every k-member subset of S is visible from a set having another configuration. The analogue is generated by replacing S by lnc S and visible by clearly visible.

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The following terminology will be used throughout the paper: conv S, cl S, int S, bdry S, and ker S will denote the convex hull, closure, interior, boundary, and kernel, respectively, for set S. Similarly, lnc S will be the set of points of local nonconvexity of S, and if S is convex, dim S will be the dimension of S. For $x \neq y$, $R(x,y)$ will represent the ray emanating from x through y, and $L(x,y)$ will be the line determined by x and y. The reader is referred to Valentine [**9**] for a discussion of the concepts.

2. The results. The following version of a result by Valentine [**10**, Corollary 2] will be needed.

LEMMA 1 (VALENTINE'S LEMMA). *Let S be a closed set in R^d. If $[x,y] \cup [y,z] \subseteq S$ and no lnc point of S lies in* $\operatorname{conv}\{x,y,z\} \sim [x,z]$, *then* $\operatorname{conv}\{x,y,z\} \subseteq S$.

The following definitions will be useful also.

DEFINITIONS. Let S be a closed set in R^d. For $q \in \operatorname{lnc} S$, define $A_q = \{x\colon q$ is clearly visible from x via $S\}$. Define

$$C_q = \bigcap\{H : H \text{ a closed halfspace with } A_q \subseteq H \text{ and } q \in \operatorname{bdry} H\}.$$

To establish the main result of the paper, it is necessary to show that for S closed and connected in R^2, $\ker S = \bigcap\{C_q \cap \operatorname{conv} S\colon q \in \operatorname{lnc} S\}$. Lemmas 2, 3, and 4 accomplish this.

LEMMA 2. *Let S be a closed set in R^d, and let $q \in \operatorname{lnc} S$. Then there is a closed halfspace H such that $A_q \subseteq H$ and $q \in \operatorname{bdry} H$.*

PROOF. Suppose on the contrary that no such halfspace H exists, to obtain a contradiction. Then $q \in \operatorname{int}\operatorname{conv} A_q$, and by Steinitz's theorem in R^d, there are $2d$ (or fewer) points $x_1, \ldots, x_{2d}$ in A_q such that $q \in \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_{2d}\}$. By standard arguments, we may select a convex neighborhood N of q, $N \subseteq \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_{2d}\}$, so that each x_i sees $S \cap \operatorname{cl} N$ via S, $1 \leq i \leq 2d$. Clearly $[x_i, q] \subseteq S$ for $1 \leq i \leq 2d$. For convenience, let $[x_i, q] \cap \operatorname{cl} N = [y_i, q]$, $1 \leq i \leq 2d$. Then for $1 \leq i, j \leq 2d$, x_j sees each point of $[y_i, q]$ via S, so $\operatorname{conv}\{y_j, y_i, q\} \subseteq S \cap \operatorname{cl} N$. By an obvious induction, $\operatorname{conv}\{y_1, \ldots, y_{2d}\} \subseteq S \cap \operatorname{cl} N$. However, it is easy to see that $q \in \operatorname{int}\operatorname{conv}\{y_1, \ldots, y_{2d}\}$, so we have a neighborhood of q in S, impossible since $q \in \operatorname{lnc} S$. We have a contradiction, our supposition must be false, and there exists a closed halfspace H having the required properties.

LEMMA 3. *Let S be a closed connected set in R^2. If $x \in \operatorname{conv} S \sim S$, then $x \notin C_{q_0}$ for at least one lnc point q_0 in S.*

PROOF. By Carathéodory's theorem in R^2, there exist points u, v, w in S such that $x \in \operatorname{conv}\{u, v, w\}$. Without loss of generality, we assume that u, v, w are noncollinear. There are two cases to consider.

Case 1. Assume for the moment that $\sim S \sim \operatorname{conv}\{u,v,w\}$ has a bounded component K, and let q be an extreme point of the compact convex set $G \equiv \operatorname{conv}(\operatorname{cl} K \cup \{x\})$ not in $\operatorname{conv}\{u,v,w\}$. It is not hard to verify that $q \in \operatorname{bdry} S$: If $q \in \sim S$, we would have a convex neighborhood of q in $\sim S \sim \operatorname{conv}\{u,v,w\}$, and this neighborhood would necessarily belong to K, impossible since $q \in \operatorname{bdry} G$. Thus $q \in S$. Since q is an extreme point of G, $q \in \operatorname{cl} K \cup \{x\} \subseteq \sim(\operatorname{int} S)$, so $q \in S \sim \operatorname{int} S = \operatorname{bdry} S$.

We assert that $q \in \operatorname{lnc} S$. Assume on the contrary that S is locally convex at q to reach a contradiction. Select a convex neighborhood N with center q such that $N \cap S$ is convex and N is disjoint from $\operatorname{conv}\{u, v, w\}$.

Let H be a supporting line to $N \cap S$ at q, and let H_1 and H_2 be the open halfspaces determined by H. Without loss of generality, we may assume that N has been chosen so that two possibilities exist (not mutually exclusive): Either $S \cap N \subseteq H \cap N$ or $N \cap \sim S$ is connected. In the first case, since $q \in \operatorname{cl} K$ and K is open, one of the halfplanes determined by H, say H_1, satisfies $H_1 \cap N \cap K \neq \emptyset$. It follows that $H_1 \cap N \subseteq K$. Hence $H \cap N \subseteq \operatorname{cl} K \subseteq G$ so that q fails to be extreme. In the second case, $N \cap \sim S \cap K \neq \emptyset$ forces $N \cap \sim S \subseteq K$ so that again $H \cap N \subseteq G$. Our assumption that $q \notin \operatorname{lnc} S$ is false, and we conclude that q is indeed an lnc point of S.

Next let M be a line supporting G at q, M_1 and M_2 corresponding open halfplanes with $G \subseteq \operatorname{cl} M_1$. We will show that $C_q \subseteq \operatorname{cl} M_2$. Assume on the contrary that there is some point in $C_q \cap M_1$. Then we may select a point $z \in A_q \cap M_1$ and a corresponding convex neighborhood N of q, with N disjoint from $\operatorname{conv}\{u, v, w\}$ and $z \notin N$, such that z sees $N \cap S$ via S. Clearly every neighborhood of q meets K, and since $K \subseteq G \subseteq \operatorname{cl} M_1$, every neighborhood of q meets $K \cap \operatorname{cl} M_1$. Hence $\operatorname{conv}((N \cap M) \cup \{z\})$ contains some point $a \in K \cap N \subseteq \operatorname{cl} M_1$. Now the ray $R(z, a)$ meets $\operatorname{bdry} K$ beyond a, say at b, with $a \in (z, b)$. Since $K \subseteq \operatorname{cl} M_1$, $b \in \operatorname{cl} M_1$, and $b \in N$. Furthermore, by our choice of q and N, $b \in \operatorname{bdry} S$, so we have $b \in \operatorname{cl} M_1 \cap N \cap S$. However, z cannot see b via S since $a \in (z, b) \cap K \subseteq (z, b) \cap \sim S$. We have a contradiction, our assumption is false, and $C_q \subseteq \operatorname{cl} M_2$.

Finally, we show that for an appropriate choice of q_0, $x \notin C_{q_0}$. There exists an edge of the simplex $\operatorname{conv}\{u, v, w\}$ such that the line it determines, call it J, separates x from some point c of K. Let J' be a line parallel to J on the c-side of J and supporting G. An extreme point of G on J' is the desired q_0. This finishes the proof in Case 1.

Case 2. Assume that $\sim S \sim \operatorname{conv}\{u, v, w\}$ has no bounded component. Let K be the component of $\sim S \cap \operatorname{conv}\{u, v, w\}$ which contains x. Since $\sim S \sim \operatorname{conv}\{u, v, w\}$ has no bounded component and S is connected, it is easy to show that K meets $\operatorname{bdry} \operatorname{conv}\{u, v, w\}$, if at all, in a segment on one edge of $\operatorname{bdry} \operatorname{conv}\{u, v, w\}$. For convenience of notation, if such a segment exists, we will assume that it lies on $[u, v]$.

Letting $G = \operatorname{conv} \operatorname{cl} K$, it is clear that G has an extreme point q not on $[u, v]$. Minor modifications in our previous argument (selecting neighborhood N of q disjoint from $[u, v]$) reveal that $q \in \operatorname{lnc} S$. Moreover, if line M supports G at q, then x lies in an open halfplane M_1 determined by M. Again adapting our earlier argument, $C_q \subseteq \operatorname{cl} M_2$. We conclude that $x \notin C_q$, finishing Case 2 and completing the proof of Lemma 3.

LEMMA 4. *Let S be a closed connected set in R^2. Then*

$$\ker S = \bigcap \{C_q \cap \operatorname{conv} S : q \in \operatorname{lnc} S\}.$$

PROOF. It is clear that $\ker S \subseteq \bigcap\{C_q \cap \operatorname{conv} S\colon q \in \operatorname{lnc} S\}$, so we need to establish only the reverse inclusion. Let x belong to $\bigcap\{C_q \cap \operatorname{conv} S\colon q \in \operatorname{lnc} S\}$. We begin by showing that $[x, q] \subseteq S$ for every lnc point q. By Lemma 3, $x \in S$. We assume that $[x, q] \not\subseteq S$ to obtain a contradiction. Then $[x, q]$ contains an open interval (a,b) in $\sim S$, and we let K denote the component of $\sim S$ which contains (a,b). Exactly two components of $K \sim L(a,b)$ contain (a,b) in their boundaries, and one of these components is in each of the open halfplanes determined by $L(a,b)$. Moreover, it is easy to see that at least one of these components, say K', must be bounded, for otherwise S could not be connected.

Let M be a line parallel to L which supports the compact set $\operatorname{conv}(\operatorname{cl} K')$, $M \neq L$. By standard arguments, M necessarily contains an extreme point q_1 of $\operatorname{conv}(\operatorname{cl} K')$. Moreover, by repeating arguments from Lemma 3, $q_1 \in \operatorname{lnc} S$. Let M_1 and M_2 denote distinct open halfplanes determined by M, with $K' \subseteq M_1$. Again employing an argument from Lemma 3, $C_{q_1} \subseteq \operatorname{cl} M_2$. But $x \in M_1$, so $x \notin C_{q_1}$. We have a contradiction, our assumption is false, and $[x,q] \subseteq S$.

Finally, we show that $x \in \ker S$. Choose point z in S to prove that $[x,z] \subseteq S$. By [**10**, Lemma 1], z sees via S some lnc point q of S. By the argument above, $[x,q] \subseteq S$, so we have a 2-path in S from x to z. If x,q,z are collinear, then the argument is finished. Hence we may assume that this is not the case.

Let w be the point of $[q,z]$ closest to z for which $\operatorname{conv}\{x,q,w\} \subseteq S$. We will show that $w = z$. Assume on the contrary that $w \neq z$, to reach a contradiction. Then by an argument involving Valentine's Lemma, there is some lnc point q_2 of $\operatorname{conv}\{x,w,z\} \cap S$ on $[x,w]$, and q_2 may be chosen closest to w. Clearly $q_2 \in \operatorname{lnc} S$. We show that $q_2 \neq w$: If $q_2 = w$, then since q_2 is an lnc point of $\operatorname{conv}\{x,w,z\} \cap S$, for every neighborhood N of w, there exist points $s \in N \cap (x,w)$ and $s' \in N \cap (z,w)$ with $[s,s'] \not\subseteq S$. But this implies that C_w lies in the convex cone at w bounded by rays $R(z,w)$ and $R(x,w)$, so x cannot lie in C_w, impossible. We conclude that $q_2 \neq w$.

Using the fact that $x \in C_{q_2}$, an argument similar to the one above reveals that $\operatorname{conv}\{q_2,w,p\} \not\subseteq S$ for any point p on $(w,z]$. By our choice of q_2, no point of $(q_2,w]$ is an lnc point for $\operatorname{conv}\{x,w,z\} \cap S$. Thus, again by an argument which uses Valentine's Lemma, for each t on $(q_2,w]$, there corresponds some t' on $(w,z]$ with $\operatorname{conv}\{t,w,t'\} \subseteq S$, and t' may be chosen closest to z. Since $\operatorname{conv}\{q_2,w,z\} \not\subseteq S$, we may select some t_0 on $(q_2,w]$ for which the corresponding t_0' is not z. We have $\operatorname{conv}\{t_0,w,t_0'\} \subseteq S$ where t_0' is as close as possible to z, so by Valentine's Lemma there must be some q_3 on $[t_0,t_0']$ such that q_3 is an lnc point for $\operatorname{conv}\{t_0,t_0',z\} \cap S$. Clearly $q_3 \in \operatorname{lnc} S$ and $q_3 \neq t_0$. Using the fact that $x \in C_{q_3}$, we may repeat an earlier argument to conclude that $q_3 \neq t_0'$. Hence $q_3 \in (t_0,t_0')$. But then C_{q_3} must lie in the closed halfplane determined by $L(t_0,t_0')$ and containing w, impossible since $x \in C_{q_3}$ and x is not in this halfplane. We have a contradiction, our original assumption must be false, and $w = z$.

We have proved that x sees z via S. Hence, $x \in \ker S$,

$$\bigcap \{C_q \cap \operatorname{conv} S : q \in \operatorname{lnc} S\} \subseteq \ker S,$$

and the sets are equal. This finishes the proof of Lemma 4.

The main results of the paper are easy consequences of our lemmas.

THEOREM 1. *Let S be a nonempty compact connected set in R^2. Then S is starshaped if and only if every* 3 *lnc points of S are clearly visible from a common point of S. The number* 3 *is best possible.*

PROOF. We may assume that S has lnc points, for otherwise S will be convex by a theorem of Tietze [8]. The necessity of the condition is obvious, so we need to establish only its sufficiency. By hypothesis, every 3 of the sets A_q, $q \in \operatorname{lnc} S$, have a nonempty intersection. Since $A_q \subseteq C_q \cap \operatorname{conv} S$, it follows that every 3 of the compact convex sets $C_q \cap \operatorname{conv} S$, $q \in \operatorname{lnc} S$, have a nonempty intersection as well. Therefore, by Helly's theorem in R^2, $\bigcap\{C_q \cap \operatorname{conv} S\colon\ q \in \operatorname{lnc} S\} \neq \emptyset$. Using Lemma 4, this intersection is exactly $\ker S$, and the theorem is proved.

To see that the number 3 in Theorem 1 is best possible, consider the following example.

EXAMPLE 1. Let S denote the set in Figure 1, with q_1, q_2, q_3 the lnc points of S and $C_i = C_{q_i}$, $1 \leq i \leq 3$. Then every 2 lnc points of S are clearly visible from a common point of S, yet $\ker S = \emptyset$.

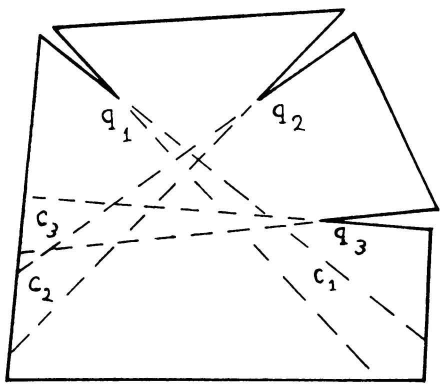

FIGURE 1

THEOREM 2. *Let S be a compact connected set in R^2, S not convex. For $k = 1$ or $k = 2$, $\dim \ker S \geq k$ if and only if for some $\epsilon > 0$, every $f(k) = \max\{3, 6, -2k\}$ points of local nonconvexity of S are clearly visible from a common k-dimensional ϵ-neighborhood in S. The number $f(k)$ is best possible.*

PROOF. Notice that since S is not convex, $\operatorname{lnc} S \neq \emptyset$. The set

$$\mathcal{B} = \{C_q \cap \operatorname{conv} S : q \in \operatorname{lnc} S\}$$

is a uniformly bounded collection of compact sets in R^2, and for some $\epsilon > 0$, every $f(k)$ members of $\mathcal{B}$ contain a common k-dimensional ϵ-neighborhood. Hence by [**2**, Lemma], $\dim \cap\{C_q \cap \operatorname{conv} S\colon q \in \operatorname{lnc} S\} \geq k$. Using Lemma 4,

$$\bigcap\{C_q \cap \operatorname{conv} S : q \in \operatorname{lnc} S\} = \ker S.$$

Thus $\dim \ker S \geq k$, and the theorem is proved.

Example 1 above shows that $f(2) = 3$ is best possible. The number $f(1) = 4$ is also best, as Example 2 reveals.

EXAMPLE 2. Let S be the set in Figure 2, with $\{q_i\colon 1 \leq i \leq 4\}$ the set of lnc points of S. Then every 3 lnc points are clearly visible from one of the segments $[z, q_i]$, $1 \leq i \leq 4$, yet $\ker S = \{z\}$.

In conclusion, notice that when set S has finitely many points of local nonconvexity, then the boundedness condition may be dropped, and we have the following result.

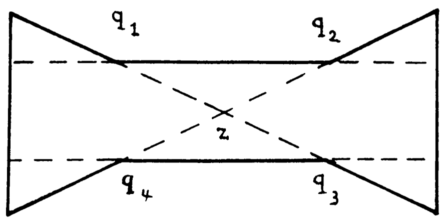

FIGURE 2

THEOREM 3. *Let S be a closed connected set in R^2 having finitely many lnc points, S not convex. Then S is starshaped if and only if every 3 lnc points of S are clearly visible from a common point of S. If $k = 1$ or $k = 2$, $\dim \ker S \geq k$ if and only if every $f(k) = \max\{3, 6 - 2k\}$ points of local nonconvexity of S are clearly visible from a common k-dimensional neighborhood in S. Each result is best possible.*

PROOF. Proof of the first statement requires the finite version of Helly's theorem, while proof of the second statement uses a result by Katchalski [5].

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