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MARKOV-TYPE PROPERTIES

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ABSTRACT. Decision problems in classes of finitely presented groups with solvable conjugacy problem, with solvable order problem, and with solvable power problem are considered. It is shown that the problem of recognizing group properties is in no real sense improved by restricting attention to these classes.

1. Introduction. In the 1950s Adjan [**1**] and Rabin [**10**] showed that many group properties are recursively unrecognizable in the class of all finitely presented (f.p.) groups. When a class is restricted, properties previously unrecognizable can become recognizable. In 1970, Collins [**4**] considered the restricted class of f.p. groups with solvable word problem, and he showed that the problem of recognizing group properties is in no real sense improved by restricting attention to this class. We restate Collins' result.

DEFINITION. A property P of groups is called *strongly-Markov* if

(i) there is a f.p. group G_1 with solvable word problem that has P,

(ii) there is a f.p. group G_2 with solvable word problem such that if G_3 is any f.p. group with solvable word problem and if G_2 is embedded in G_3, then G_3 does not have P.

THEOREM (COLLINS). *If P is a strongly-Markov property, then there is a recursive class $\mathcal{C}_P$ of finite presentations such that each Π of $\mathcal{C}_P$ has solvable word problem and P is unrecognizable in $\mathcal{C}_P$.*

This paper is concerned with variations of Collins' result. Classes of finite presentations with solvable conjugacy problem, with solvable order problem, and with solvable power problem will be studied.

The results of this paper are included in the author's doctoral thesis, which was written under the direction of W. W. Boone at the University of Illinois. The author would like to thank W. W. Boone and P. E. Schupp for their help.

2. Statements and proofs of results. We begin with a definition.

DEFINITION. Let λ be a decision problem for groups. A property P of groups is called λ-*Markov* if

(i) there is a f.p. group G_1 with solvable λ-problem that has P,

(ii) there is a f.p. group G_2 with solvable λ-problem such that if G_3 is a f.p. group with solvable λ-problem and if G_2 is embedded in G_3, then G_3 does not have P. We call G_1 the *positive witness* and G_2 the *negative witness* of P.

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The analogues of Collins' result will be proved for λ equal to conjugacy (C), order (O), and power (P). In contrast to Collins' proof, the results of this paper will be proved using a small cancellation construction.

Let G, G_1, and G_2 be f.p. groups. We define (in a manner similar to that of Schupp [**11**]) an *Adjan-Rabin construction with respect to* (G_1, G_2) to be a procedure θ associating to every word W on the generators of G a presentation $\theta(G, W)$ with the following properties.

(i) If $W = 1$ in G, then $\theta(G, W) \cong G_1$.

(ii) If $W \neq 1$ in G, then G_2 is embedded in $\theta(G, W)$.

Similarly, we define an *order Adjan-Rabin construction with respect to* (G_1, G_2) to be a procedure θ associating to every word W on the generators of G a presentation $\theta(G, W)$ with the following properties.

(i) If $o(W)$, the order of W, in G is nonzero, then $\theta(G, W) \cong G_1$.

(ii) If $o(W)$ is zero, then G_2 is embedded in $\theta(G, W)$.

That is, we switch from dependence on the word problem to dependence on the order problem. In either case, if λ is a decision problem for groups, we say that θ *preserves* λ if, whenever G has solvable λ-problem, $\theta(G, W)$ has solvable λ-problem for all words W of G; and we say that θ is *totally effective* if $\{\theta(G, W) | W$ is a word of $G\}$ is recursive for all finite presentations G.

PROPOSITION 1. *Let P be a λ-Markov property with positive witness G_1 and negative witness G_2. Suppose*

(a) *there is a recursive class $\mathcal{C}$ of f.p. groups such that each element of $\mathcal{C}$ has solvable λ-problem but $\mathcal{C}$ does not have uniformly solvable word problem;*

(b) *there is a totally effective Adjan-Rabin construction θ with respect to (G_1, G_2) that preserves λ.*

Then there is a recursive class $\mathcal{D}$ of f.p. groups each of which has solvable λ-problem and such that P is unrecognizable in $\mathcal{D}$.

PROOF. We claim that $\mathcal{D} = \{\theta(G, W) | G$ is in $\mathcal{C}$ and W is a word of $G\}$ is the desired class. Since each G has solvable λ-problem and θ preserves λ, each element of $\mathcal{D}$ has solvable λ-problem. Since θ is an Adjan-Rabin construction, $\theta(G, W)$ has P if and only if $W = 1$ in G. Since $\mathcal{C}$ does not have uniformly solvable word problem, P is unrecognizable in $\mathcal{D}$.

PROPOSITION 2. *Let P be a λ-Markov property with positive witness G_1 and negative witness G_2. Suppose*

(a) *there is a f.p. group G_0 with solvable λ-problem ($\lambda \neq 0$), solvable word problem, and unsolvable order problem;*

(b) *there is a totally effective order Adjan-Rabin construction θ with respect to (G_1, G_2) that preserves λ.*

Then there is a recursive class $\mathcal{D}$ of f.p. groups each of which has solvable λ-problem and such that P is unrecognizable in $\mathcal{D}$.

PROOF. Take $\mathcal{D} = \{\theta(G_0, W) | W$ is a word of $G_0\}$. The verification that $\mathcal{D}$ satisfies the conditions stated above is similar to the verification in the proof of Proposition 1. Note that we require G_0 to have solvable word problem as well as unsolvable order problem to guarantee the nonexistence of an algorithm to determine whether or not $o(W)$ is zero in G_0.

The advantage of Proposition 2 over Proposition 1 is that only a single group with solvable λ-problem is needed instead of a class of groups. Before proceeding to the analogues of Collins' result, we need certain lemmas.

LEMMA 1. *There exists a recursive class $\mathcal{C}$ of f.p. groups such that each element of $\mathcal{C}$ has solvable conjugacy problem but $\mathcal{C}$ does not have uniformly solvable word problem.*

PROOF. Collins [**5**] has verified that, for a given Thue system $\mathcal{T}$, the class of f.p. groups $\{G_{\mathcal{T},\varphi} | \varphi$ is a word of $\mathcal{T}\}$ constructed by Boone satisfies the following conditions.

(i) The conjugacy problem for $G_{\mathcal{T},\varphi}$ is Turing equivalent to the word problem for $\mathcal{T}$.

(ii) If φ and ψ are words of $\mathcal{T}$, then $\varphi = \psi$ in $\mathcal{T}$ if and only if $W_1(\psi) = W_2(\psi)$ in $G_{\mathcal{T},\varphi}$, where $W_1(\psi)$ and $W_2(\psi)$ are certain words of $G_{\mathcal{T},\varphi}$.

Let $\mathcal{E} = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \dots\}$ be a recursive class of Thue systems such that each $\mathcal{T}_i$ has solvable word problem but the word problem for $\mathcal{E}$ is not uniformly solvable.

For the existence of such a class, see Boone and Rogers [**3**]. We claim that $\mathcal{C} = \{G_{\mathcal{T}_i,\varphi} | i \in \omega, \varphi$ is a word on $\mathcal{T}_i\}$ meets the requirements of the lemma. By (i), each $G_{\mathcal{T}_i,\varphi}$ has solvable conjugacy problem. By (ii), $\mathcal{C}$ does not have uniformly solvable word problem.

LEMMA 2. *There exists a recursive class $\mathcal{C}$ of f.p. groups such that each element of $\mathcal{C}$ has solvable order problem but $\mathcal{C}$ does not have uniformly solvable word problem.*

PROOF. Let $\theta = \{G_\Gamma | \Gamma$ is a special word of $T_*\}$ be as defined by Boone [**2**] and Collins [**4**]. Boone and Collins have shown that each G_Γ has solvable word problem but θ does not have uniformly solvable word problem. Each group G_Γ is constructed from the free group $\langle x, y; \emptyset\rangle$ by a sequence of HNN-extensions. Since $\langle x, y; \emptyset\rangle$ has solvable order problem, one can see that each G_Γ has solvable order problem by showing certain membership problems to be solvable.

We are now ready to prove the analogues of Collins' result.

THEOREM 1. *If P is a C-Markov property, then there is a recursive class $\mathcal{D}$ of f.p. groups such that each element of $\mathcal{D}$ has solvable conjugacy problem but P is unrecognizable in $\mathcal{D}$.*

PROOF. Let G_1 be the positive witness and G_2 be the negative witness of P. By Proposition 1 and Lemma 1, it is enough to show that there is an Adjan-Rabin construction θ with respect to (G_1, G_2) that preserves conjugacy. The construction given below is a slight modification of a construction due to Schupp [**11**].

Suppose that G is an arbitrarty f.p. group and W is a word of G. We may assume that G is generated by $\{g_1, \dots, g_m\}$ and G_2 is generated by $\{g_{m+1}, \dots, g_n\}$. Let

$$Q = G * G_2, \quad K = Q * \langle a; a^3 = 1\rangle * \langle b; b^3 = 1\rangle,$$
$$D_W = \langle K; M\rangle, \quad \theta(G, W) = D_W * G_1,$$

where

$$M = \{(Wa)^{97}, (Wb)^{97}, g_j(ab)^{80(3j-3)+1}ab^2(ab)^{80(3j-3)+2} \cdot ab^2 \cdots (ab)^{80(3j-2)} \ (j = 1, \dots, n)\}.$$

First note that if $W = 1$ in G, then $\theta(G, W)$ is isomorphic to G_1. It is not difficult to see that if $W \neq 1$ in G and if R_W is the symmetrized subset of K generated by M, then R_W satisfies the small cancellation condition $C'(\frac{1}{6})$. Therefore, if $W \neq 1$ in G, G_2 is embedded in $\theta(G, W)$ (see [**8**, p. 278]). Therefore, θ is a totally effective Adjan-Rabin construction with respect to (G_1, G_2).

It remains to show that θ preserves conjugacy. So assume that G has solvable conjugacy problem. The result follows trivially if $W = 1$ in G, so assume $W \neq 1$ in G. It is clear that K has solvable conjugacy problem. To show that D_W, and hence $\theta(G, W)$, has solvable conjugacy problem, it suffices to verify that R_W satisfies $C'(\frac{1}{6})$ and Condition J (see [**8**, Theorem V. 9.6]). We have already noted that R_W satisfies $C'(\frac{1}{6})$. Recall Condition J.

CONDITION J. If r is in R_W, then r is not conjugate to r^{-1} in K.

Define α: $K \to \langle a; a^3 = 1\rangle$ by $\alpha(a) = a$, $\alpha(b) = 1$, and $\alpha(g_j) = 1$ for $j = 1, \ldots, n$. Then α is a homomorphism. Suppose that r is any cyclic permutation of $(Wa)^{97}$ and c is any word of K. If $r = c^{-1}r^{-1}c$, then $\alpha(r) = \alpha(c^{-1}r^{-1}c)$, and so, $a^{97} = a^{-97}$ in $\langle a; a^3 = 1\rangle$. Therefore, r is not conjugate to r^{-1} in K. The arguments for the remaining elements of R_W are similar.

THEOREM 2. *If P is an O-Markov property, then there is a recursive class $\mathcal{D}$ of f.p. groups such that each element of $\mathcal{D}$ has solvable order problem and P is unrecognizable in $\mathcal{D}$.*

PROOF. The proof is entirely analogous to the proof of Theorem 1, using Lemma 2 instead of Lemma 1 and McCool [**9**] instead of Theorem V. 9.6 of [**8**].

The following theorem can be proved in a manner analogous to that for Theorem 2. However, the method will be changed in order to exhibit an order Adjan-Rabin construction.

THEOREM 3. *If Q is a P-Markov property, then there is a recursive class $\mathcal{D}$ of f.p. groups such that each element of $\mathcal{D}$ has solvable power problem and Q is unrecognizable in $\mathcal{D}$.*

PROOF. Let G_1 be the positive witness and G_2 the negative witness for Q. We verify the hypotheses of Proposition 2 for λ equal to P. Collins [**6**] has shown the existence of a f.p. group with solvable word and power problems and unsolvable order problem. We define an order Adjan-Rabin construction θ with respect to (G_1, G_2). For an arbitrary f.p. group G and for each word W on the generators of G, let

$\bar{Q} = \langle r, s, t, u; s^{-1}rs = r^2, t^{-1}st = s^2, u^{-1}tu = t^2, r^{-1}ur = u^2\rangle$,
$H_w = \langle G * \bar{Q}; W = r\rangle$,
$J_W = G_2 * H_W$,
$E_W = J_W * \langle a; a^3 = 1\rangle * \langle b; b^3 = 1\rangle$,
$D_W = \langle E_W; N_W\rangle$,
$\theta(G, W) = D_W * G_1$,

where

$$N_W = \{(Wa)^{97}, (Wb)^{97}, g_j(ab)^{80(3j-3)+1}ab^2 \cdots (ab)^{80(3j-2)} (j = 1, \ldots, n)\}$$

and J_W is generated by $\{g_1, \ldots, g_n\}$.

We must verify that θ is an order Adjan-Rabin construction and that θ preserves the power problem.

Higman [7] has shown that $\langle \bar{Q}; r^k = 1 \rangle$ is trivial if $k \neq 0$. Therefore, it is easy to see that $o(W) \neq 0$ in G implies that D_W is trivial and $\theta(G, W)$ is isomorphic to G_1. Hence, $\theta(G, W)$ has solvable power problem and has property Q if $o(W) \neq 0$.

We claim that if $o(W) = 0$ in G_1 then

(i) J_W has solvable power problem if G has solvable power problem; and

(ii) if R_W is the symmetrized subset of E_W generated by N_W, then R_W satisfies the small cancellation condition $C'(\frac{1}{6})$.

Assuming the claim, (ii) yields that if $o(W) = 0$, then G_2 is embedded in $\theta(G, W)$—completing the verification that θ is an order Adjan-Rabin construction. By (i) and a result of McCool [9], $\theta(G, W)$ has solvable power problem if G has solvable power problem completing the verification that θ preserves the power problem.

It remains to prove the claim.

(i) It is straightforward to verify that $\bar{Q}$ has solvable power problem. Since $o(W) = o(r) = 0$, H_W is a free product with amalgamation. Since G and $\bar{Q}$ have solvable power problem and the membership problems for $\{W^i | i \in \omega\}$ in G and $\{r^i | i \in \omega\}$ in $\bar{Q}$ are solvable, H_W has solvable power problem. Hence, J_W has solvable power problem.

(ii) Since $o(W) = 0$ in G, H_W is a free product with amalgamation and $W \neq 1$ in J_W. Claim (ii) then follows as in the proof of Theorem 1.

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