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ON THE SURJECTIVITY OF $\Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F)$

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ABSTRACT. If E and F are vector bundles over a compact connected Moisezon complex manifold X with canonical bundle K and $E \otimes K^{-1}$ and $F \otimes K^{-1}$ are positive in the sense of Nakano then $\Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F)$ is surjective.

0. Introduction. Let X be a complex analytic space, $\mathcal{E}$ and $\mathcal{F}$ be analytic sheaves over X, Γ be the functor of global sections and

$$\Gamma(\mathcal{E}) \otimes \Gamma(\mathcal{F}) \to \Gamma(\mathcal{E} \otimes \mathcal{F})$$

be the natural morphism of vector spaces. The question is whether the morphism is surjective.

On a compact Riemann surface of genus $g \geq 2$ it is easy to construct counterexamples: line bundles E and F with the morphism in question failing to be surjective when either $\deg(E \otimes K^{-1}) < 0$ or $\deg(F \otimes K^{-1}) < 0$, where K is the canonical line bundle. For then $\Gamma(E)$ (or $\Gamma(F)$) is trivial while $\Gamma(E \otimes F)$ can be nontrivial.

Thus the question becomes: is the morphism surjective when we have a canonical sheaf $\mathcal{K}$ with inverse $\mathcal{K}^{-1}$ with $\mathcal{E} \otimes \mathcal{K}^{-1}$ and $\mathcal{F} \otimes \mathcal{K}^{-1}$ both positive in some sense? The answer, in a fairly general case, is yes.

THEOREM. *If X is a compact connected complex Moisezon manifold with vector bundles E and F such that, for the canonical line bundle K, $E \otimes K^{-1}$ and $F \otimes K^{-1}$ are both positive in the sense of Nakano then the morphism*

$$\Gamma(E) \otimes \Gamma(F) \to \Gamma(E \otimes F)$$

is surjective.

The proof uses a long exact sequence, a Künneth formula, and a vanishing theorem. The vanishing theorem needs the positivity conditions. The long exact sequence is constructed below. The Künneth formula is that proved by L. Kaup [**1**] for coherent analytic sheaves. The vanishing theorem is proved by Grauert and Riemenschneider [**2**] for coherent analytic sheaves on Moisezon spaces.

1. The proof. Consider the diagonal map d and projections p_1, p_2 related to the product space as shown below.

$$\begin{tikzcd} & & X \\ X \arrow[r, "d"] & X \times X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & & X \end{tikzcd}$$

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Consider E to be a vector bundle over the first factor and F to be a vector bundle over the second factor. Then p_1^*E and p_2^*F are vector bundles over $X \times X$ and $p_1^*E \otimes p_2^*F$ is the vector bundle usually denoted by $E\hat{\otimes}F$. Then

$$d^*(E\hat{\otimes}F) = E \otimes F.$$

Let $\mathcal{I}_{dX}$ be the sheaf of ideals in the structure sheaf $\mathcal{O}_{X\times X}$ for the irreducible submanifold $d(X)$. Then the sheaf

$$\hat{\mathcal{O}}_X \underset{\text{def}}{=} \mathcal{O}_{X\times X}/\mathcal{I}_{dX}$$

has $d^*(\hat{\mathcal{O}}_X) = \mathcal{O}_X$.

Now consider the sheaves of germs of holomorphic sections $\mathcal{O}_{X\times X}(E\hat{\otimes}F)$ and

$$\hat{\mathcal{O}}_X(E\hat{\otimes}F) \underset{\text{def}}{=} d_*^0(\mathcal{O}_X(E \otimes F))$$

where d_*^0 is the 0th direct image functor. Also consider the sheaf

$$\mathcal{I}_{dX} \otimes \mathcal{O}_{X\times X}(E\hat{\otimes}F) \underset{\text{def}}{=} \mathcal{I}_{dX}(E\hat{\otimes}F)$$

of germs of sections of $E\hat{\otimes}F$ which vanish along $d(X)$.

We then have the exact sequence

$$0 \to \mathcal{I}_{dX}(E\hat{\otimes}F) \to \mathcal{O}_{X\times X}(E\hat{\otimes}F) \to \hat{\mathcal{O}}_X(E\hat{\otimes}F) \to 0$$

which gives rise to the cohomology long exact sequence

$$\begin{aligned} 0 &\to H^0(X \times X, \mathcal{I}_{dX}(E\hat{\otimes}F)) \to H^0(X \times X, \mathcal{O}_{X\times X}(E\hat{\otimes}F)) \\ &\to H^0(X \times X, \hat{\mathcal{O}}_X(E\hat{\otimes}F)) \to H^1(X \times X, \mathcal{I}_{dX}(E\hat{\otimes}F)) \to \cdots. \end{aligned}$$

The sheaves are all over $X \times X$.

By Kaup's Künneth formula,

$$H^0(X \times X, \mathcal{O}_{X\times X}(E\hat{\otimes}F)) \simeq H^0(X, \mathcal{O}_X(E)) \otimes H^0(X, \mathcal{O}_X(F))$$

through the projection maps. Furthermore, since

$$\hat{\mathcal{O}}_X(E\hat{\otimes}F) = d_*^0(\mathcal{O}_X(E \otimes F))$$

we get

$$H^0(X \times X, \hat{\mathcal{O}}_X(E\hat{\otimes}F)) \simeq H^0(X, \mathcal{O}_X(E \otimes F)).$$

Thus with $H^0 = \Gamma$, we see that the theorem follows as soon as we have

$$H^1(X \times X, \mathcal{I}_{dX}(E\hat{\otimes}F)) = 0.$$

Here is where the vanishing theorem of Grauert and Riemenschneider is used.

The vanishing theorem says:

If Y is an irreducible compact connected Moisezon space and S is a quasipositive torsion free coherent analytic sheaf on Y then

$$H^q(Y, S \cdot K_Y) = 0, \quad \text{for } q \geq 1,$$

where K_Y is the canonical sheaf on Y. The main idea of quasipositivity requires positivity in the sense of Nakano only on a dense open subset of the set $R(Y, S)$ of regular points in Y over which S is locally free. (See the reference for complete definitions.)

We apply this result to the sheaf

$$\mathcal{I}_{dX} \otimes \mathcal{O}_{X\times X}(E\hat{\otimes}F) \otimes \mathcal{O}_{X\times X}(K_{X\times X}^{-1})$$

over $Y = X \times X$ where $K_{X\times X}$ is the canonical line bundle over $X \times X$. The quasipositivity necessary is to be the positivity off of dX. This is implied by the positivity assumptions of the theorem because of the following.

First, $\mathcal{I}_{dX}$ is trivial off of dX so the positivity of

$$\mathcal{O}_{X\times X}(E\hat{\otimes}F) \otimes \mathcal{O}_{X\times X}(K_{X\times X}^{-1})$$

on all of $X \times X$ will suffice.

Second, $K_{X\times X} = p_1^* K_X \otimes p_2^* K_X$ so the sheaf is just

$$\mathcal{O}_{X\times X}((E\hat{\otimes}F) \otimes (K_X\hat{\otimes}K_X)^{-1})$$

which is

$$\mathcal{O}_{X\times X}((E \otimes K_X^{-1})\hat{\otimes}(F \otimes K_X^{-1})).$$

This is

$$d_*^0(\mathcal{O}_X(E \otimes K_X^{-1}) \otimes \mathcal{O}_X(F \otimes K_X^{-1})),$$

which is easily seen to be positive by the assumptions of the theorem.

This finishes the proof.

2. Remarks. Of course the proof also tells the difference between $\Gamma(E) \otimes \Gamma(F)$ and $\Gamma(E \otimes F)$, it is just

$$H^0(X \times X, \mathcal{I}_{dX}(E\hat{\otimes}F)).$$

The proof also works for more general cases, in particular for coherent analytic sheaves instead of for vector bundles since the Künneth formula and the vanishing theorem hold for them. Any generalization of the vanishing theorem which allows the taking of the inverse of the canonical sheaf over analytic spaces would allow the theorem to hold over analytic spaces instead of manifolds.

References

1. L. Kaup, *Eine Künnethformel für Fréchetgarben*, Math. Z. **97** (1967), 158–168.
2. H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.

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