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ANALYTIC CONTINUATION ON COMPLEX LINES

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ABSTRACT. The following extension theorem is proved. Let $\Omega \subset \mathbf{C}$ be an open set containing Δ, the open unit disc in $\mathbf{C}$, and the point 1. Suppose that f is holomorphic on B, the open unit ball of $\mathbf{C}^N$, let $x \in \partial B$ and assume that for all $y \in \partial B$ in a neighborhood of x the function $\varsigma \to f(\varsigma y)$, holomorphic on Δ, continues analytically into Ω. Then f continues analytically into a neighborhood of x.

On B, the open unit ball of $\mathbf{C}^N$, $N > 1$, there are holomorphic functions f with the property that for any complex line Λ that meets B the function $f|(B \cap \Lambda)$, holomorphic on $B \cap \Lambda$, continues analytically across no point of $\partial B \cap \Lambda$, see [**2**]. This shows that a holomorphic function f on B that continues analytically across no point of ∂B, can be highly noncontinuable. The complementary question to be considered is what continuability properties can such a function have on complex lines. For instance, if f continues analytically across no point of ∂B, can it happen that for every $y \in \partial B$ the function $\varsigma \to f(\varsigma y)$, holomorphic on Δ, the open unit disc in $\mathbf{C}$, continues analytically across $\partial\Delta$? This motivated us to prove the extension theorem below. As a consequence, we prove that if f is a holomorphic function on B that continues analytically across no point of ∂B then the set of $y \in \partial B$ for which $\varsigma \to f(\varsigma y)$ continues analytically across a point $\partial\Delta$, is of the first category, which answers the last question in the negative.

THEOREM. *Let f be a holomorphic function on B, let $x \in \partial B$ and let $S \subset \partial B$ be a neighborhood of x. Suppose that $R > 1$ and let $\Omega \subset \mathbf{C}$ be an open set containing Δ and the segment $[1, R]$. Assume that for every $y \in S$ the function $\varsigma \to f(\varsigma y)$, holomorphic on Δ, continues analytically into Ω. Then f continues analytically into an open set containing the segment $\{tx\colon 1 \le t \le R\}$.*

To prove the theorem we need the following consequence of the deep Hartogs's lemma [**1**, p. 139]:

LEMMA. *Let D_1, D_2 be domains in $\mathbf{C}^{n-1}$, $\mathbf{C}$ respectively. If F is a function on $D_1 \times D_2$ such that F is holomorphic on $D_1 \times P$ for an open set $P \subset D_2$ and such that for each fixed $z \in D_1$ the function $w \to F(z, w)$ is holomorphic on D_2 then F is holomorphic on $D_1 \times D_2$.*

PROOF OF THEOREM. With no loss of generality, assume that S is open in ∂B and that Ω is connected. By the assumption, for each $y \in S$ there is a holomorphic function ϕ_y on Ω such that $f(\varsigma y) = \phi_y(\varsigma)$ $(\varsigma \in \Delta)$. By the identity theorem

$$\tilde{f}(\varsigma y) = \phi_y(\varsigma) \qquad (\varsigma \in \Omega, y \in S)$$

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gives a well-defined extension $\tilde{f}$ of f to

$$Q = B \cup \left[\bigcup_{y \in S} \{\varsigma y \colon \varsigma \in \Omega\} \right].$$

We will complete the proof by showing that $\tilde{f}$ is holomorphic in a neighborhood of $\{tx \colon 1 \leq t \leq R\}$. We assume, without loss of generality, that $x = (0, 0, \ldots, 0, 1)$. Since Q is a neighborhood of $\{tx \colon 0 \leq t \leq R\}$ one can choose an open connected neighborhood $U \subset \Omega$ of $[0, R]$ and a δ, $0 < \delta < 2^{-1/2}$ such that for every $z = (z_1, z_2, \ldots, z_{N-1})$ satisfying $\|z\| = (|z_1|^2 + |z_2|^2 + \cdots + |z_{N-1}|^2)^{1/2} < \delta$ and for every $w \in U$ we have

$$(wz, w) \in Q, \tag{1}$$

$$w(1 + \|z\|^2)^{1/2} \in \Omega, \tag{2}$$

$$(1 + \|z\|^2)^{-1/2}(z, 1) \in S. \tag{3}$$

For every $(z, w) \in \mathbf{C}^{N-1} \times \mathbf{C}$ such that $(wz, w) \in Q$ define

$$F(z, w) = \tilde{f}(wz, w).$$

We now apply the lemma to the function F. Let $D_1 = \{z \in \mathbf{C}^{N-1} \colon \|z\| < \delta\}$, $D_2 = U$ and $P = \{w \in \mathbf{C} \colon |w| < \delta\} \cap U$. Since $\tilde{f}$ is well defined in Q, it follows by (1) that F is defined in $D_1 \times D_2$. Furthermore, since $\tilde{f}$ is holomorphic in B, F is holomorphic on the open set $\{(z, w) \in \mathbf{C}^N \colon \|wz\|^2 + |w|^2 < 1\}$ which contains $D_1 \times P$ and hence the function $(z, w) \to F(z, w)$ is holomorphic on $D_1 \times P$. Now let $z \in D_1$. If $w \in D_2$ then $F(z, w) = \tilde{f}(wz, w) = \phi_q(w(1 + \|z\|^2)^{1/2})$ where $q = (1 + \|z\|^2)^{-1/2}(z, 1)$. By (3), $q \in S$, so by the assumption ϕ_q is holomorphic on Ω. By (2), it follows that the function $w \to F(z, w)$ is holomorphic on D_2 for all $z \in D_1$. The facts imply that the hypothesis of the lemma are satisfied and so F is holomorphic on

$$V = \{(z, w) \colon \|z\| < \delta, w \in U\}. \tag{4}$$

Let $U_1 \subset U$ be an open neighborhood of $[1, R]$ whose closure does not contain 0, i.e. $\sup_{w \in U_1} \frac{1}{|w|} = \tau < \infty$. Choose $\eta > 0$, $\eta\tau < \delta$, and put $W = \{(z, w) \colon \|z\| < \eta,\ w \in U_1\}$. The map $(z, w) \to \phi(z, w) = (\frac{1}{w} z, w)$, holomorphic on W, satisfies $\phi(W) \subset V$, which, together with (4) implies that $(z, w) \to \tilde{f}(z, w) = F(\frac{1}{w} z, w) = (F \circ \phi)(z, w)$ is holomorphic on W, a neighborhood of $\{tx \colon 1 \leq t \leq R\}$. This completes the proof.

REMARK 1. In the special case when Ω is a disc centered at 0 the extension result follows from a theorem of Forelli [**3**, p. 61]. Specifically, if f is holomorphic on B and if for every $y \in \partial B$ there is an r_y such that $\varsigma \to f(\varsigma y)$ continues analytically into the disc $|\varsigma| < r_y$ then by Forelli's theorem, f continues analytically into the interior of the set $\bigcup_{y \in \partial B} \{\varsigma y \colon |\varsigma| < r_y\}$.

REMARK 2. As seen from the proof of the theorem the main idea was to use a nonsingular holomorphic map in a neighborhood of x that locally maps the family of complex lines through the origin into a family of parallel lines, to apply Hartogs's lemma on parallel lines and then apply the inverse of this map. It is clear that the same idea works in the setting of more general domains in place of B.

COROLLARY. *Let f be a holomorphic function on B that continues analytically across no point of ∂B. Then the set of all those $y \in \partial B$ for which the function $\varsigma \to f(\varsigma y)$ continues analytically across a point of $\partial\Delta$, is of the first category.*

PROOF. Let $z_n \in \partial\Delta$ be a dense sequence and let r_n, $0 < r_n < 1/2$, converge to zero. For each m, $n \in \mathbf{N}$ let

$$\Omega_{m,n} = \{z \in \mathbf{C}\colon |z| < 1 - r_n\} \cup \{z \in \mathbf{C}\colon |z - z_m| < 2r_n\}.$$

Let S be the set of all $y \in \partial B$ for which the function $\varsigma \to f(\varsigma y)$ continues analytically across a point of $\partial\Delta$. For every $y \in S$ there are some m, $n \in \mathbf{N}$ such that $\varsigma \to f(\varsigma y)$ continues analytically into a neighborhood of $\overline{\Omega_{m,n}}$. The set S can be written as a union

$$S = \bigcup_{m,n,p=1}^{\infty} S_{m,n,p}$$

where, for each m, n, p, $S_{m,n,p} = \{y \in S\colon \varsigma \to f(\varsigma y)$ continues analytically into $\Omega_{m,n}$ and the continuation is bounded on $\Omega_{m,n}$ by $p\}$. We will complete the proof by showing that $S_{m,n,p}$ is nowhere dense for every m, n, p.

By the theorem no $S_{m,n,p}$ contains an open set. So to prove that $S_{m,n,p}$ is nowhere dense for every m, n, p it suffices to show that $S_{m,n,p}$ is compact for every m, n, p. To do this, fix m, n, p and let $y_k \in S_{m,n,p}$ converge to y. We have to show that $y \in S_{m,n,p}$.

For each k there is a function ϕ_k, analytic and bounded by p on $\Omega_{m,n}$ such that $\phi_k(\varsigma) = f(\varsigma y_k)$ $(|\varsigma| \leq 1/2)$. Since f is continuous on B it follows that $\phi_k(\varsigma)$ converges to $f(\varsigma_y)$ uniformly on $\{\varsigma\colon |\varsigma| \leq 1/2\}$. By a normal family argument it follows that $\phi_k(\varsigma)$ converges on $\Omega_{m,n}$ to an analytic function $\phi(\varsigma)$, bounded on $\Omega_{m,n}$ by p. Since $\phi(\varsigma) = f(\varsigma y)$ $(|\varsigma| \leq 1/2)$ it follows that $\varsigma \to f(\varsigma y)$ continues analytically to $\Omega_{m,n}$ and that the continuation is bounded on $\Omega_{m,n}$ by p. Consequently, $y \in S_{m,n,p}$. This completes the proof.

ADDED IN PROOF. It has been pointed out to the authors by J. Siciak that the basic theorem also appears in a paper by M. Downarowicz, *Analytic Continuation of Series of Homogeneous Polynomials of n Complex Variables*, Zeszyty Naukowe Uniwersytetu Jagiellonskiego, 1975.

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